Homework 1

Analysis and Linear Algebra I
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Indian Institute of Science

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1. From first principles, prove the following:

   (a) $A \subseteq A \cup B$

   (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

2. Prove that one of the following formulas is always right and that the other is sometimes wrong.

   (a) $A \setminus (B \setminus C) = (A \setminus B) \cup C$

   (b) $A \setminus (B \cup C) = (A \setminus B) \setminus C$

3. Using only the field axioms for $\mathbb{R}$, prove that $a(b - c) = ab - ac$ for $a, b, c \in \mathbb{R}$ and $\frac{b}{a} = ba^{-1}$ for $a, b \in \mathbb{R}, a \neq 0$.

4. Using the field axioms and order axioms for $\mathbb{R}$, prove that, for $a, b, c \in \mathbb{R}$, if $a > b$ and $c > 0$, then $ac > bc$.

5. Using the field axioms and order axioms for $\mathbb{R}$, prove that there is no $x \in \mathbb{R}$ which satisfies $x^2 + 1 = 0$.

6. Prove that if $a, x, y \in \mathbb{R}$ satisfy
$$a \leq x \leq a + \frac{y}{n}$$
   for every integer $n \geq 1$, then $x = a$.

7. If $x, y \in \mathbb{R}$ such that $x < y$, then prove that there exists $z \in \mathbb{R}$ such that $x < z < y$.

8. If $x \in \mathbb{Q}, x \neq 0$ and $y \in \mathbb{R} \setminus \mathbb{Q}$, then prove that $x + y, \frac{x}{y}, \frac{y}{x} \in \mathbb{R} \setminus \mathbb{Q}$.

9. Prove that there is no rational number whose square is 2.