# Characters of classical groups twisted by roots of unity 

A Dissertation<br>submitted in partial fulfillment of the requirements for the award of the<br>degree of<br>

by
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## Declaration

I hereby declare that the work reported in this thesis is entirely original and has been carried out by me under the supervision of Prof. Arvind Ayyer at the Department of Mathematics, Indian Institute of Science, Bangalore. I further declare that this work has not been the basis for the award of any degree, diploma, fellowship, associateship or similar title of any University or Institution.

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01-01-00-10-11-18-1-16328

Indian Institute of Science,
Bangalore,
July, 2023.

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## Dedicated to

my parents $\mathfrak{F}$ teachers.

## Acknowledgments

First and foremost, I would like to express my gratefulness towards Prof. Arvind Ayyer, my Ph.D. advisor, for his guidance, unfailing support, patience, and constant encouragement during the entire journey of my Ph.D. His insightful academic suggestions not only helped me in my research work but also motivated me in improving myself. Furthermore, I convey my profound thanks to him for introducing me to the beautiful world of algebraic combinatorics. Also, I am highly obliged to him for going through all the drafts of my research articles including this thesis meticulously and helping me to improve them.

The Ph.D. fellowship came from NBHM - an Indian government scholarship programme to foster the development of higher mathematics in the country, through which I registered for the doctoral program at the Department of Mathematics, Indian Institute of Science (IISc), Bangalore.

I thank the members of my comprehensive panel, Prof. Sunil Chandran (Department of Computer Science and Automation, IISc), Prof. B. Sury (Indian Statistical Institute, Bangalore), Prof. R. Venkatesh (Department of Mathematics, IISc) and Prof. Kaushal Verma (Department of Mathematics, IISc) for their remarks during my comprehensive examination. I want to thank the chair of our department, Prof. A. K. Nandakumaran and all faculty members for their availability whenever approached. I would like to extend my sincere thanks and gratefulness to the faculty members of the Department of Mathematics at IISc for offering a wide variety of courses. I really benefited a lot from many of these courses. I am also thankful to all the Mathematics Department staff at IISc for their help in carrying out all the complex administrative formalities smoothly during my stay at IISc. Needless to say about the beautiful academic environment and facilities of the Mathematics Department for carrying out my research.

I would like to express my gratitude and sincere thanks to Prof. Dipendra Prasad and my advisor Prof. Arvind Ayyer for arranging my research visit to IIT Bombay. I extend my sincere thanks to Prof. Dipendra Prasad and Prof. Murali K. Srinivasan for insightful academic discussions during the research visit. I am thankful to my colleagues/friends Dibyendu Biswas and Chayan Karmakar for helping me and making my visit pleasant.

I want to thank my friends who have some way or the other, helped me gain the knowledge and skills needed to build up this thesis, Dr. Subhajit Ghosh (Bar-Ilan University, Israel) and Dr. Krishna Teja (ISI Bangalore) for mathematical discussions and proofreading the articles, and Nimisha Pahuja (Mathematics, IISc) for helping me before presentations. Special thanks to Dr. Prakash Chandra Arya (Earth Science, IISc) and Prashant Kumar (IAP, IISc) for their selfless support. It is my pleasure to have a
wonderful class of colleagues/friends at the Department of Mathematics, IISc. I would like to express my thanks to all of them.

Last but not least, I would like to express my gratitude to my parents, sisters, and brother for their continuous motivation and moral support during my Ph.D. Without their support, it would not have been possible to finish this work.

## Abstract

This thesis focuses on the study of specialized characters of irreducible polynomial representations of infinite families of complex classical Lie groups. We study various specializations where the characters are evaluated at elements twisted by roots of unity. The details of the results are as follows.

Throughout the thesis, we fix an integer $t \geqslant 2$ and a primitive $t^{\text {th }}$ root of unity $\omega$. We first consider the irreducible characters of representations of the classical groups $\mathrm{GL}_{t n}, \mathrm{SO}_{2 t n+1}, \mathrm{Sp}_{2 t n}$ and $\mathrm{O}_{2 t n}$, evaluated at elements $\omega^{k} x_{i}$ for $0 \leqslant k \leqslant t-1$ and $1 \leqslant i \leqslant n$. The case of $\mathrm{GL}_{t n}$ was considered by D. J. Littlewood (AMS press, 1950) and independently by D. Prasad (Israel J. Math., 2016). In each case, we characterize partitions for which the character value is nonzero in terms of what we call $z$-asymmetric partitions, where $z$ is an integer which depends on the group. This characterization turns out to depend on the $t$-core of the indexed partition. Furthermore, if the character value is nonzero, we prove that it factorizes into characters of smaller classical groups. We also give product formulas for general $z$-asymmetric partitions and $z$-asymmetric $t$-cores, and show that there are infinitely many $z$-asymmetric $t$-cores for $t \geqslant z+2$.

We extend the above results to the groups $\mathrm{GL}_{t n+m}(0 \leqslant m \leqslant t-1), \mathrm{SO}_{2 t n+3}$, $\mathrm{Sp}_{2 t n+2}$ and $\mathrm{O}_{2 t n+2}$ evaluated at similar specializations. For the $\mathrm{GL}_{t n+m}$ case, we set the first $t n$ elements to $\omega^{j} x_{i}$ for $0 \leqslant j \leqslant t-1$ and $1 \leqslant i \leqslant n$ and the remaining $m$ to $y, \omega y, \ldots, \omega^{m-1} y$. For the other three families, we take the same specializations but with $m=1$. Our motivation for studying these are the conjectures of Wagh and Prasad (Manuscripta Math., 2020) relating the irreducible representations of $\operatorname{Spin}_{2 n+1}$ and $\mathrm{SL}_{2 n}$, $\mathrm{SL}_{2 n+1}$ and $\mathrm{Sp}_{2 n}$ as well as $\operatorname{Spin}_{2 n+2}$ and $\mathrm{Sp}_{2 n}$.

The hook Schur polynomials are the characters of covariant and contravariant irreducible representations of the general linear Lie superalgebra. These are a supersymmetric analogue of the characters of irreducible polynomial representations of the general linear group. Finally, we consider similarly specialized skew hook Schur polynomial hs $\mathrm{h}_{\lambda / \mu}$ evaluated at $\omega^{k} x_{i} / \omega^{\ell} y_{j}$, for $0 \leqslant k, \ell \leqslant t-1,1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. We characterize the skew shapes $\lambda / \mu$ for which the polynomial vanishes and prove that the nonzero
polynomial factorizes into smaller skew hook Schur polynomials. Then we give a combinatorial interpretation of $\mathrm{hs}_{\lambda / \mu}\left(1, \omega^{d}, \ldots, \omega^{d(t n-1)} / 1, \omega^{d}, \ldots, \omega^{d(t m-1)}\right)$, for all divisors $d$ of $t$, in terms of ribbon supertableaux.

For certain combinatorial objects, the number of fixed points under a cyclic group action turns out to be the evaluation of a nice function at the roots of unity. This is known as the cyclic sieving phenomenon (CSP) and has been the focus of several studies. We use the combinatorial interpretation for the above skew hook Schur polynomial to prove the CSP on the set of semistandard supertableaux of shape $\lambda / \mu$ for odd $t$. Using a similar proof strategy, we give a complete generalization of a result of Lee-Oh (Electron. J. Combin., 2022) for the CSP on the set of skew SSYT conjectured by Alexandersson-Pfannerer-Rubey-Uhlin (Forum Math. Sigma, 2021).

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## Chapter 1

## Introduction

In this thesis, we study characters of irreducible representations of complex classical Lie groups with different specializations using combinatorial techniques. The characters of irreducible representations of the classical families of groups, namely the general linear, symplectic and orthogonal groups are important families of symmetric Laurent polynomials indexed by integer partitions or half-partitions. In particular, the characters of the general linear groups are the Schur polynomials, which are extremely wellstudied [17, 77, 89, 110]. These families of Laurent polynomials with specialized indeterminates satisfy nontrivial relations, which are not well understood from the point of view of representation theory (see, for instance [10, 11, 12, 30, 66]). Littlewood [74] considered the Schur polynomials in $t n$ variables, for $t \geqslant 2$ a fixed positive integer, specialized to $\left(\exp (2 \pi \iota k / t) x_{j}\right)_{0 \leqslant k \leqslant t-1,1 \leqslant j \leqslant n}$. Motivated by a celebrated result of Kostant [65], Prasad [90] also considered the Schur polynomial with the same specialization independently and evaluated the character of irreducible representations of $\mathrm{GL}_{t n}(\mathbb{C})$ on the subgroup $\mathrm{GL}_{t}(\mathbb{C})^{n} \rtimes \mathbb{Z}_{n}$, which sits naturally inside $\mathrm{GL}_{t n}(\mathbb{C})$ at elements of the subgroup which have projection a fixed generator $\sigma$ of $\mathbb{Z}_{n}$. They showed that such a specialized character is nonzero if and only if the corresponding $t$-core is empty, and if it is nonzero, it factors into characters indexed by the $t$-quotients; see Chapter 2 for the definitions. We extend their results in a few different directions in the next few chapters. We also prove the cyclic sieving phenomenon on the set of tableaux and supertableaux in Chapter 6 .

We begin this chapter by surveying related works in Section 1.1. Later, we give an overview of the organisation and the layout of the thesis in Section 1.2,

### 1.1 Brief literature review

This section provides a brief overview of the theory of classical groups, including their origin, significance, and notable results pertaining to their representation theory. The complex classical Lie groups, also simply known as the classical groups, are four infinite families of Lie groups that, along with the five exceptional finite groups, make up the complete classification of simple Lie groups. The history of Lie groups dates back to the 19th century, with the emergence of the theory of quadratic forms and their associated transformations. During the late 19th century, Sophus Lie, a Norwegian mathematician, extensively investigated continuous transformation groups, which are now known as Lie groups, both geometrically and analytically. Lie's work ultimately led to the development of the modern theory of Lie groups, which bears his name 50.

Lie made a significant contribution to mathematics by uncovering that continuous transformation groups could be better comprehended by linearizing them and examining their corresponding generating vector fields. The generators follow a linearized form of the group law, known as the commutator bracket, and possess the characteristics of what is now referred to as a Lie algebra [29, 48, 51]. This laid the groundwork for the study of the symmetries of mathematical objects, including the classical groups. The classification of the simple Lie algebras over complex numbers, which led to the discovery of the classical groups, was accomplished by Wilhelm Killing and Élie Cartan.

The first classical group to be introduced was the orthogonal group, which was studied extensively by mathematicians such as Cartan and Killing in the late 19th and early 20th centuries. The unitary groups and symplectic groups were introduced later [40, 50]. In his renowned volume [119], Weyl provided a description of the structure of these groups, which applies to both general fields and $p$-adic fields, with more intricate details presented for the latter. Weyl's seminal book played a significant role in establishing Lie group theory as a basic field of research in mathematics building on the work of Lie, Killing, Cartan and the invariant theorists of the nineteenth century. Since that time, progress in Lie theory has advanced at an impressive pace. See [42, 45, 47] for a modern introduction to the field.

Classical groups have numerous applications in various fields, including physics, chemistry and coding theory. In physics, especially in quantum mechanics, they are used to describe symmetries of physical systems and the behaviour of elementary particles [101, 118]. The application of group representations also proved to be an immensely valuable tool for spectroscopy, as well as for providing quantum-mechanical interpretations of chemical bonds [120]. In coding theory, error-correcting codes are used to protect digital data from errors that may occur during transmission, storage, or signal process-
ing, and the theory of classical groups is used to construct error-correcting codes [25, 27]. These groups have provided an empirical basis for large parts of algebra [32] and the study of their representation theory has also led to significant breakthroughs in pure mathematics, including algebraic geometry and number theory. Also, see [117] for a nice survey on the geometry of classical groups over finite fields and its applications.

Now we recall some key results pertaining to the representation theory of classical groups over the field of complex numbers. Since the complex classical Lie groups are linear groups, their finite-dimensional representations are tensor representations by Weyl's construction. Each irreducible polynomial representation is labelled by a partition or half-partition, which encodes its structure and properties [40].

The irreducible polynomial representation of the general linear group $\mathrm{GL}_{n}(\mathbb{C})$ indexed by a partition $\lambda$ has a basis indexed by semistandard Young tableaux of shape $\lambda$ with entries from $\{1,2, \ldots, n\}$. The number of semistandard tableaux of shape $\lambda$ is therefore equal to the dimension of the representation, and this is given by Weyl's dimension formula [119]. In a similar fashion, semistandard symplectic and orthogonal tableaux of shape $\lambda$, which index bases for the irreducible polynomial representations associated with $\lambda$ for $\operatorname{Sp}(2 n)$ and $\mathrm{SO}(m)$ have been introduced by various authors (see, for instance, [60, 61, 92, 109]). El Samra and King [99] manipulated Weyl's dimension formula to count the number of semistandard symplectic and odd orthogonal tableaux of shape $\lambda$ in terms of hook lengths and contents and to produce formula for the dimension of the representation. Recently, Amdeberhan, Andrews and Ballantine [8] gave combinatorial interpretations of analogous expressions involving hook-lengths and symplectic or orthogonal contents.

The character of an irreducible polynomial representation of $\mathrm{GL}_{n}(\mathbb{C})$ corresponding to the partition $\lambda$ is the Schur function $s_{\lambda}\left(x_{1}, x_{2}, \ldots, x_{n}\right)$. Stanley [104] showed that its principal specialization $s_{\lambda}\left(1, q, \ldots, q^{n-1}\right)$ could be expressed as a product involving the hook lengths and contents of the boxes in the diagram for $\lambda$. This gives a generating function for the semistandard tableaux of shape $\lambda$ with entries in $\{1,2, \ldots, n\}$ and in particular, taking $q=1$ yields a formula for the number of such tableaux. Similar specializations in the symplectic and orthogonal cases have been studied by Koike 63] and by Campbell and Stokke [26]. Furthermore, Schur polynomials evaluated at roots of unity and their powers have been considered in [77, 94].

In a different direction, it was shown in [30] that the Schur polynomial for a rectangular partition in $2 n$ variables specialized to the last $n$ variables being reciprocals of the first $n$ variables becomes a product of two other classical characters. In some cases, this is the product of a symplectic and an even orthogonal character, and in others, it is the product of two odd orthogonal characters. As an application, a factorization
theorem for rhombus tilings of a hexagon is given, which has an equivalent formulation in terms of plane partitions. Similar factorization results were obtained in [11 for socalled double staircase partitions, i.e. partitions of the form $(k, k, k-1, k-1, \ldots, 1,1)$ or $(k, k-1, k-1, \ldots .1,1)$. This kind of factorization was generalized in [10] for a large class of partitions and further, to skew-Schur functions, i.e. induced characters, in [12].

Next, we discuss supersymmetric Schur functions, a supersymmetric analogue of the Schur functions. The supersymmetric Schur functions, also known as the hook Schur functions, were introduced by Berele and Regev [19] in their study of Lie superalgebras. Lie superalgebras or $\mathbb{Z}_{2}$-graded Lie algebras are Lie algebras of Lie supergroups, whose function algebras are algebras with commuting and anticommuting variables [20, 21, [57].

Lie superalgebras and their representations continue to play an important role in physics in the context of supersymmetries relating particles of different statistics [31]. Lie superalgebras have applications in quantum mechanics [7], conformal field theory [33], string theory [37], nuclear physics [15], solvable lattice models [14, 103], supergravity [9], spin systems [44] and quantum systems [98]. Their affine extensions or $q$-deformations have also been studied [7, 33] to understand physical systems.

Representation theory of Lie superalgebras differs from the corresponding theory of Lie algebras in a non-linear manner. Fueled by the physicists' keen interest in the subject, Kac constructed a theory of Lie superalgebras and gave a classification of classical Lie superalgebras [57, 100]. Then he proceeded to the problem of classifying all finitedimensional irreducible representations of the classical Lie superalgebras [56]. He derived a character formula closely analogous to the Weyl character formula for a class of irreducible representations of simple Lie superalgebras [56]. The characters of covariant and contravariant irreducible representations of $\mathfrak{g l}(m \mid n)$ are identified with supersymmetric Schur functions [19, 34], where the corresponding supersymmetric Schur function is labelled by a single partition $\lambda$. But for the mixed tensor irreducible representations, the corresponding supersymmetric function is labelled by a composite partition. The problem of obtaining a character formula for the remaining irreducible representations has been the subject of intensive investigation [86, 108, 112, 113, 114 .

### 1.2 Organisation of the thesis

We are motivated by the work of Littlewood [74] that studies specialized irreducible classical characters of the general linear group. We generalize the result to the characters of other classical groups. Our goal is, on the one hand, to characterize the partitions for which the specialized irreducible classical character is zero and, on the other hand, to
prove that the non-zero character factorizes into characters of smaller groups. Presently we don't understand these results at the level of the representations of classical groups. The organization of this thesis is the following:

In Chapter 2, we discuss the preliminaries necessary for this thesis. We briefly recall various bases of the ring of symmetric and supersymmetric functions and irreducible characters of classical groups. We also discuss combinatorial objects associated with the symmetric and supersymmetric functions. This is followed by an introduction to the cyclic sieving phenomenon.

In Chapter 3, we generalize Littlewood's results to other classical groups $\mathrm{Sp}_{2 t n}$, $\mathrm{SO}_{2 t n+1}$ and $\mathrm{O}_{2 t n}$ and obtain factorizations for their characters under the same specialization as that of Littlewood. We use Cauchy-type determinant formulas for these characters and study the beta sets of partitions. For the general linear group, there is only one possible value of the $t$-core for which the twisted character is nonzero, namely the empty partition. For the other classical characters, there are many possible values of the $t$-core for which the character is nonzero. We will show that these are $t$-cores which can be written in Frobenius coordinates as $(\alpha \mid \alpha+z)$, where the value of $z$ depends on the group, and which we call $z$-asymmetric partitions. Further, we give product formulas for general $z$-asymmetric partitions and $z$-asymmetric $t$-cores. Lastly, we show that there are infinitely many $z$-asymmetric $t$-cores for $t \geqslant z+2$.

In Chapter 4. we give new proofs of factorization results proved in Chapter 3 using Jacobi-Trudi type identities. Recently using a similar proof strategy, Albion [2] lifted all the factorization results to the level of universal characters.

In Chapter 5, we extend the factorization results to the groups $\mathrm{GL}_{t n+m}(0 \leqslant m \leqslant$ $t-1), \mathrm{SO}_{2 t n+3}, \mathrm{Sp}_{2 t n+2}$ and $\mathrm{O}_{2 t n+2}$ evaluated at similar specializations: (1) for the $\mathrm{GL}_{t n+m}$ case, we set the first $t n$ elements to $\omega^{j} x_{i}$ for $0 \leqslant j \leqslant t-1$ and $1 \leqslant i \leqslant n$ and the remaining $m$ to $y, \omega y, \ldots, \omega^{m-1} y$; (2) for the other three families, the same specializations as above but with $m=1$. For the general linear group, we prove that there are finitely many $t$-cores for which the twisted character is nonzero. For the other classical characters, we characterize partitions for which the character value is nonzero in terms of what we call $\left(z_{1}, z_{2}, k\right)$-asymmetric partitions, where $z_{1}, z_{2}$ and $k$ are integers which depend on the group. Lastly, we prove that there are infinitely many $t$-core partitions for which these characters are nonzero.

In Chapter 6, we consider the specialized skew hook Schur polynomial and consequently get the factorization of skew Schur polynomial with the same specialization as that of Littlewood. We also give a combinatorial proof of the skew Schur factorization result for $t=2$. Then we prove the cyclic sieving phenomenon on the set of semistandard tableaux and supertableaux of shape $\lambda / \mu$.

## Chapter 2

## Preliminaries

The purpose of this chapter is on the one hand to fix some notation and terminology, and on the other hand, to introduce the ring of symmetric and supersymmetric functions briefly. As most of the objects considered turn out to be indexed by partitions, we will introduce them in Section 2.1. Section 2.2 is dedicated to the ring of symmetric functions and well-known bases for this ring. We define Schur polynomials and the skew Schur polynomials, special classes of symmetric functions, in Section 2.3. In Section 2.4, we consider the characters of irreducible polynomial representations of classical groups. Next in Section 2.5, we briefly discuss the ring of supersymmetric functions. In Section 2.6, we define ribbon tableau and supertableau, generalizing the definitions of tableau and supertableau defined in Section 2.1 and Section 2.5 respectively. Lastly in Section 2.7, we discuss some results related to the cyclic sieving phenomenon.

### 2.1 Partitions

Recall that a partition $\lambda$ is a weakly decreasing sequence of nonnegative integers $\lambda=$ $\left(\lambda_{1}, \ldots, \lambda_{\ell}\right)$. The non-zero elements $\lambda_{i}$ are called the parts of $\lambda$. The length of a partition $\lambda$, denoted $\ell(\lambda)$ is the number of parts of $\lambda$, and the sum of the parts is the weight of $\lambda$, denoted by $|\lambda|$. By $a+\lambda$, for $a \in \mathbb{N}$, we will mean the partition $\left(a+\lambda_{1}, \ldots, a+\lambda_{\ell}\right)$. For a partition $\lambda$ and an integer $\ell$ such that $\ell(\lambda) \leqslant \ell$, define the beta-set of $\lambda$ to be a strict partition $\beta(\lambda) \equiv \beta(\lambda, \ell)=\left(\beta_{1}(\lambda, \ell), \ldots, \beta_{\ell}(\lambda, \ell)\right)$ where $\beta_{i}(\lambda, \ell)=\lambda_{i}+\ell-i$. We will write $\beta(\lambda)$ whenever $\ell$ is clear from the context.

A partition $\lambda$ can be represented pictorially as a Young diagram, whose $i^{\prime}$ th row contains $\lambda_{i}$ left-justified boxes. We will use the so-called English notation where the first
row is on top. For example, the Young diagram of $(4,2,2,1)$ is


For a partition, $\lambda$, the conjugate partition, denoted $\lambda^{\prime}$, is the partition whose Young diagram is obtained by transposing the Young diagram of $\lambda$. A partition in which no part occurs more than once is called a strict partition. The (Frobenius) rank of a partition $\lambda$, denoted $\operatorname{rk}(\lambda)$, is the largest integer $k$ such that $\lambda_{k} \geqslant k$. The Frobenius coordinates of $\lambda$ is a pair of strict partitions, denoted $(\alpha \mid \beta)$, of length at most $\operatorname{rk}(\lambda)$ given by $\alpha_{i}=\lambda_{i}-i$ and $\beta_{j}=\lambda_{j}^{\prime}-j$. For example, the Frobenius coordinates of our running example $(4,2,2,1)$ in 2.1.1) are $(3,0 \mid 3,1)$.

Recall that for partitions $\lambda$ and $\mu$, we write $\mu \subset \lambda$ to mean that the Young diagram of $\lambda$ contains the Young diagram of $\mu$, which is the same as $\mu_{i} \leqslant \lambda_{i}$, for all $i \geqslant 1$. The skew shape $\lambda / \mu$ is the set-theoretic difference $\lambda \backslash \mu$. For example, the Young diagram of the skew shape $(4,2,2,1) /(2,1)$ is


A path in a skew shape $\lambda / \mu$ is a sequence $x_{0}, x_{1}, \ldots, x_{k}$ of squares in $\lambda / \mu$ such that $x_{i-1}$ and $x_{i}$ have a common side, for $1 \leqslant i \leqslant k$. A subdiagram $\phi$ of $\lambda / \mu$ is said to be connected if any two squares in $\phi$ can be connected by a path in $\phi$. A border strip or ribbon is a connected subdiagram of the Young diagram of $\lambda$ which contains no $2 \times 2$ block of squares. Therefore, successive rows and columns of a border strip overlap by exactly one box. The length of a border strip $\zeta$ is the total number of boxes it contains and its height, denoted ht $(\zeta)$, is defined to be one less than the number of rows it occupies. For example,

is a border strip of length 6 and height 3. A domino is a border strip of length 2. The
head of a border strip is the rightmost box in its top row.
We fix $t$ to be an integer greater than or equal to 2 . Now we first define the $t$-core and $t$-quotient of a partition following [75]. There are many equivalent definitions (see for instance [41, 49, 53, 116]). We then recall Macdonald's criterion to find the $t$-core and $t$-quotient using the beta set.

Definition 2.1. The $t$-core of the partition $\lambda$, denoted $\operatorname{core}_{t}(\lambda)$, is the partition obtained by successively removing border strips of length $t$ from the Young diagram of $\lambda$.

In the example given in (2.1.1), we see that $\operatorname{core}_{2}((4,2,2,1))=(2,1)$ after three border strip removals. The idea of removing border strips to get the $t$-core of a partition goes back to Nakayama [81]. For example, the only 2-cores are staircase shapes, i.e. partitions of the form $(k, k-1, \ldots, 1,0), k \in \mathbb{N}$.

For a cell $c=(i, j)$ in (the Young diagram of) $\lambda$, the hook length is given by $h_{c}=$ $\lambda_{i}-i+\lambda_{j}^{\prime}-j+1$, which is the total number of cells in its row to the right and those in its column below it including the cell itself. The content of $c$ is $j-i$. The arm (resp. leg) of $c$ is the rightmost (resp. bottommost) cell in its row (resp. column). For example, the hook lengths and contents of the running example are


Definition 2.2. The $t$-quotient of $\lambda$ is a $t$-tuple of partitions denoted quo $_{t}(\lambda)=\left(\lambda^{(0)}, \ldots\right.$, $\lambda^{(t-1)}$ ) obtained using the Young diagram of $\lambda$. The $(i+1)$ 'th element of this tuple is obtained by taking all cells $c$ whose hook length is divisible by $t$, and whose arm has content congruent to $i(\bmod t)$. It is a nontrivial fact that this collection of cells forms a Young subdiagram of $\lambda$. The corresponding partition is $\lambda^{(i)}$.

From (2.1.2), we see that $\operatorname{quo}_{2}((4,2,2,1))=((2),(1))$. Macdonald [77] defines the $t$-core and $t$-quotient alternately using the beta-set and we recall this construction. Let $\lambda$ be a partition with $\ell(\lambda) \leqslant \ell$. For $0 \leqslant i \leqslant t-1$, let $n_{i}(\lambda) \equiv n_{i}(\lambda, \ell)$ be the number of parts of $\beta(\lambda)$ congruent to $i(\bmod t)$ and $\beta_{j}^{(i)}(\lambda), 1 \leqslant j \leqslant n_{i}(\lambda)$ be the $n_{i}(\lambda)$ parts of $\beta(\lambda)$ congruent to $i(\bmod t)$ in decreasing order.

Proposition 2.3 ([77, Example I.1.8]). Let $\lambda$ be a partition of length at most $\ell$.

1. The $\ell$ numbers $t j+i$, where $0 \leqslant j \leqslant n_{i}(\lambda)$ and $0 \leqslant i \leqslant t-1$, are all distinct. Arrange them in descending order, say $\tilde{\beta}_{1}>\cdots>\tilde{\beta}_{\ell}$. Then the $t$-core of $\lambda$ has
parts $\left(\operatorname{core}_{t}(\lambda)\right)_{i}=\tilde{\beta}_{i}-\ell+i$. Thus, $\lambda$ is a $t$-core if and only if these $\ell$ numbers $t j+i$, where $0 \leqslant j \leqslant n_{i}(\lambda)$ and $0 \leqslant i \leqslant t-1$ form its beta-set $\beta(\lambda)$.
2. The parts $\beta_{j}^{(i)}(\lambda)$ may be written in the form $t \tilde{\beta}_{j}^{(i)}+i, 1 \leqslant j \leqslant n_{i}(\lambda)$, where $\tilde{\beta}_{1}^{(i)}>\cdots>\tilde{\beta}_{n_{i}(\lambda)}^{(i)} \geqslant 0$. Let $\lambda_{j}^{(i)}=\tilde{\beta}_{j}^{(i)}-n_{i}(\lambda)+j$, so that $\lambda^{(i)}=\left(\lambda_{1}^{(i)}, \ldots, \lambda_{n_{i}(\lambda)}^{(i)}\right)$ is a partition. Then the $t$-quotient $\operatorname{quo}_{t}(\lambda)$ of $\lambda$ is a cyclic permutation of $\lambda^{\star}=$ $\left(\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(t-1)}\right)$. The effect of changing $\ell$ is to permute the $\lambda^{(j)}$ cyclically so that $\lambda^{\star}$ should perhaps be thought of as a 'necklace' of partitions.

Remark 2.4. We note that Macdonald's definition of the $t$-quotient is not identical to that of Definition 2.2, but is equal up to a cyclic shift. In particular, if $\mathrm{quo}_{t}(\lambda)=$ $\left(\lambda^{(0)}, \ldots, \lambda^{(t-1)}\right)$ and $m$ increases by 1 in Proposition 2.3, the new $t$-quotient will be $\left(\lambda^{(t-1)}, \lambda^{(0)}, \ldots, \lambda^{(t-2)}\right)$.

A tableau (or semistandard Young tableau) $T$ of shape $\lambda / \mu$ is a filling of the Young diagram of $\lambda / \mu$ in such a way that the numbers increase strictly down each column and weekly from left to right along each row. The sequence $\left(c_{1}(T), c_{2}(T), \ldots\right)$, where $c_{i}(T)$ be the number of occurences of $i$ in $T$, is called the weight of $T$. Denote the set of semistandard Young tableaux of shape $\lambda / \mu$ filled with numbers in $\{1, \ldots, k\}$ by $\operatorname{SSYT}_{k}(\lambda / \mu)$.

Example 2.5. Consider $\lambda=(4,2,2,1)$ and $\mu=(2,1)$. Then the following figure illustrates a tableau of shape $(4,2,2,1) /(2,1)$ and of weight $(4,1,1)$.


There exists a natural partial order on the set of partitions of $m$ called the dominance order, denoted $\vDash$. For two partitions $\mu$ and $\lambda$ of weight $m$, we write $\mu \vDash \lambda$ if $\mu_{1}+\cdots+\mu_{i} \leqslant$ $\lambda_{1}+\cdots+\lambda_{i}$ for all $i \geqslant 1$. In that case, we say that $\lambda$ dominates $\mu$. Let $K_{\lambda, \mu}$ be the number of tableaux of shape $\lambda$ and weight $\mu$. Then $K_{\lambda, \mu}$ is positive if and only if $\lambda$ dominates $\mu$ in the dominance partial order. Also $K_{\mu, \mu}=1$. See [77, Chapter 1] for more details.

### 2.2 The ring of symmetric functions

Consider the ring $\mathbb{Z}\left[x_{1}, \ldots, x_{n}\right]$ of polynomials in $n$ independent variables with integer coefficients. A polynomial in the ring $\mathbb{Z}\left[x_{1}, \ldots, x_{n}\right]$ is symmetric if it is invariant under
the action of permuting the variables. Let $\Lambda\left[x_{1}, \ldots, x_{n}\right]$ denote the subring of symmetric functions, which is graded by degree: we have

$$
\Lambda\left[x_{1}, \ldots, x_{n}\right]=\bigoplus_{k \geqslant 0} \Lambda^{k}\left[x_{1}, \ldots, x_{n}\right]
$$

where $\Lambda^{k}\left[x_{1}, \ldots, x_{n}\right]$ consists of the homogeneous symmetric polynomials of degree $k$, together with the zero polynomial. See [77, 96, 104] for more details. Several bases indexed by partitions are defined on $\Lambda\left[x_{1}, \ldots, x_{n}\right]$. We define some of them after setting up a few notations.

We will use $n$ for a fixed positive integer and let $X=\left(x_{1}, \ldots, x_{n}\right)$ be a tuple of commuting indeterminates. For any integer $j$, we set $X^{j}=\left(x_{1}^{j}, \ldots, x_{n}^{j}\right)$, and for $a \in \mathbb{R}$, set $a X=\left(a x_{1}, \ldots, a x_{n}\right)$. Define $\bar{x}=1 / x$ for an indeterminate $x$ and write $\bar{X}=\left(\bar{x}_{1}, \ldots, \bar{x}_{n}\right)$. For each $n$-tuple $\alpha=\left(\alpha_{1}, \ldots, \alpha_{n}\right) \in \mathbb{Z}_{\geqslant 0}^{n}$, we denote the monomial $x_{1}^{\alpha_{1}} \ldots x_{n}^{\alpha_{n}}$ by $X^{\alpha}$.

Let $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}\right)$ be a partition of length at most $n$. The monomial symmetric function indexed by $\lambda$ is defined as

$$
\begin{equation*}
m_{\lambda}(X):=\sum_{\alpha} X^{\alpha}, \tag{2.2.1}
\end{equation*}
$$

summed over all distinct permutations $\alpha$ of $\lambda$. As $\lambda$ runs through all partitions of length at most $n$, the monomial symmetric functions $m_{\lambda}(X)$, form a $\mathbb{Z}$-basis for the ring $\Lambda[X]$. The elementary symmetric function $e_{\lambda}(X)$ indexed by $\lambda$ is defined as

$$
\begin{equation*}
e_{\lambda}(X):=\prod_{i=1}^{n} e_{\lambda_{i}}(X), \tag{2.2.2}
\end{equation*}
$$

where

$$
e_{r}(X):=\sum_{1 \leqslant i_{1}<i_{2}<\cdots<i_{r} \leqslant n} x_{i_{1}} x_{i_{1}} \ldots x_{i_{r}}=m_{\left(1^{r}\right)}(X) \text { for } r \geqslant 1 \text { and } e_{0}(X):=1 .
$$

The functions $e_{r}(X)$ are algebraically independent over $\mathbb{Z}$ and the set of all $e_{\lambda}(X)$ with $\ell\left(\lambda^{\prime}\right) \leqslant n$ form a $\mathbb{Z}$-basis for $\Lambda[X]$. The complete symmetric function $h_{\lambda}(X)$ indexed by $\lambda$ is defined as

$$
\begin{equation*}
h_{\lambda}(X):=\prod_{i=1}^{n} h_{\lambda_{i}}(X), \tag{2.2.3}
\end{equation*}
$$

where

$$
h_{r}(X):=\sum_{1 \leqslant i_{1} \leqslant i_{2} \leqslant \cdots \leqslant i_{r} \leqslant n} x_{i_{1}} x_{i_{1}} \ldots x_{i_{r}}=\sum_{|\lambda|=r} m_{\lambda}(X) \text { for } r \geqslant 1 \text { and } h_{0}(X):=1 .
$$

The set of all $h_{\lambda}(X)$ with $\ell\left(\lambda^{\prime}\right) \leqslant n$ also form a $\mathbb{Z}$-basis for $\Lambda[X]$. The power sum symmetric function $p_{\lambda}(X)$ indexed by $\lambda$ is defined as

$$
\begin{equation*}
p_{\lambda}(X):=\prod_{i=1}^{n} p_{\lambda_{i}}(X), \tag{2.2.4}
\end{equation*}
$$

where $p_{r}(X):=\sum_{i=1}^{r} x_{i}^{r}=m_{(r)}(X)$ for $r \geqslant 1$ and $p_{0}(X):=1$. It is convenient to define $e_{r}(X), h_{r}(X)$ and $p_{r}(X)$ to be zero for $r<0$. It is shown in [77, Chapter 1, Section 2] that $r h_{r}(X)=\sum_{s=1}^{r} p_{s}(X) h_{r-s}(X)$. So $\mathbb{Q}\left[p_{1}(X), \ldots, p_{r}(X)\right]=\mathbb{Q}\left[h_{1}(X), \ldots, h_{r}(X)\right]$. Since the complete symmetric functions $h_{r}(X)$ are algebraically independent over $\mathbb{Z}$, and hence also over $\mathbb{Q}$, the $p_{r}(X)$ are also algebraically independent over $\mathbb{Q}$. So, the $p_{\lambda}(X)$ form a $\mathbb{Q}$-basis of $\Lambda[X]$. But they do not form a $\mathbb{Z}$-basis of $\Lambda[X]$; for example, $h_{2}(X)=\frac{1}{2}\left(p_{1}(X)^{2}+p_{2}(X)\right)$ does not have integer coefficients. We note the following generating function identities:

$$
\begin{align*}
& \sum_{r \geqslant 0} e_{r}(X) q^{r}=\prod_{i=1}^{n}\left(1+x_{i} q\right),  \tag{2.2.5}\\
& \sum_{r \geqslant 0} h_{r}(X) q^{r}=\prod_{i=1}^{n} \frac{1}{1-x_{i} q}, \tag{2.2.6}
\end{align*}
$$

and

$$
\begin{equation*}
\sum_{r \geqslant 1} p_{r}(X) q^{r-1}=\sum_{i=1}^{n} \frac{x_{i}}{1-x_{i} q} . \tag{2.2.7}
\end{equation*}
$$

We can also consider symmetric polynomials in countably many independent variables $x_{1}, x_{2}, \ldots$. Denote the ring thus obtained by $\Lambda$. Note that the elements of $\Lambda$ are no longer polynomials, they are formal infinite sums of monomials. Let $m_{\lambda}, e_{\lambda}, h_{\lambda}$ and $p_{\lambda}$ be the corresponding monomial, elementary, complete and power sum symmetric functions in infinitely many variables $x_{1}, x_{2}, \ldots$ On the ring of symmetric functions $\Lambda$, a ring homomorphism is defined by: $\check{\omega}: \Lambda \rightarrow \Lambda$ which maps $e_{r}$ to $h_{r}$, for all $r \geqslant 0$. This homomorphism is an involution [77], i.e. $\breve{\omega}^{2}$ is the identity map. The involution $\check{\omega}$ maps a power sum onto a scalar multiple of itself:

$$
\check{\omega}\left(p_{\lambda}\right)=\epsilon_{\lambda} p_{\lambda} \text { with } \epsilon_{\lambda}=(-1)^{|\lambda|-\ell(\lambda)} \text {. }
$$

Using this involution, a fifth $\mathbb{Z}$-basis of $\Lambda$ can be defined for any partition $\lambda$, namely

$$
f_{\lambda}=\check{\omega}\left(m_{\lambda}\right) .
$$

These elements are called the forgotten symmetric functions, as there is no simple direct
description.
We now define the standard Hall inner product $\langle u, v\rangle$, a $\mathbb{Z}$-valued bilinear form $\Lambda$. The basis elements $\left(h_{\lambda}\right)$ and $\left(m_{\lambda}\right)$ are defined to be dual of each other with respect to this inner product:

$$
\left\langle h_{\lambda}, m_{\mu}\right\rangle=\delta_{\lambda, \mu},
$$

for all partitions $\lambda, \mu$ of length at most $n$, where $\delta_{\lambda, \mu}$ is the Kronecker delta. Then

$$
\left\langle e_{\lambda}, f_{\mu}\right\rangle=\delta_{\lambda, \mu},
$$

for all partitions $\lambda, \mu$ of length at most $n$, since the involution $\check{\omega}$ is an isometry, i.e $\langle\check{\omega}(u), \check{\omega}(v)\rangle=\langle u, v\rangle$. Also one can prove that

$$
\left\langle p_{\lambda}, p_{\mu}\right\rangle=\delta_{\lambda, \mu} z_{\lambda},
$$

for all partitions $\lambda, \mu$ of length at most $n$, where $z_{\lambda}=\prod_{i \geqslant 1} i^{m_{i}} m_{i}$ ! and $m_{i}$ is the number of parts of $\lambda$ equal to $i$. See [77, Chapter 1, Section 4] for more details.

Recall that we have fixed $t$ to be an integer greater than or equal to 2 . Let $\omega$ be a primitive $t^{\prime}$ th root of unity, i.e. $\omega^{t}=1$ and $\omega^{s} \neq 1$ for any $s<t$. If $\lambda=\left(\lambda_{1}, \ldots, \lambda_{\ell}\right)$ is a partition such that $t$ divides $\lambda_{i}$, for all $i$, then we write $\frac{\lambda}{t}$ for the partition $\left(\frac{\lambda_{1}}{t}, \ldots, \frac{\lambda_{\ell}}{t}\right)$.

Theorem 2.6. For a partition $\lambda$ of length at most tn, the specialized elementary, complete and power sum symmetric functions, $e_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right), h_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)$ and $p_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)$ respectively are given by

$$
\begin{align*}
& e_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=\left\{\begin{array}{lll}
0 & \lambda_{i} \not \equiv 0 & (\bmod t) \text { for some } i, \\
e_{\frac{\lambda}{t}}\left((-1)^{t-1} X^{t}\right) & \text { otherwise },
\end{array}\right.  \tag{2.2.8}\\
& h_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)= \begin{cases}0 & \lambda_{i} \not \equiv 0 \quad(\bmod t) \text { for some } i, \\
h_{\frac{\lambda}{t}}\left(X^{t}\right) & \text { otherwise },\end{cases}  \tag{2.2.9}\\
& p_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)= \begin{cases}0 & \lambda_{i} \neq 0 \quad(\bmod t) \text { for some } i, \\
t p_{\frac{\lambda}{t}}\left(X^{t}\right) & \text { otherwise } .\end{cases} \tag{2.2.10}
\end{align*}
$$

Proof of Theorem 2.6. By 2.2.5, we see that the generating function for the required
elementary symmetric function is

$$
\begin{aligned}
\sum_{r \geqslant 0} e_{r}\left(X, \omega X, \ldots, \omega^{t-1} X\right) q^{r}=\prod_{i=1}^{n}(1 & \left.+x_{i} q\right)\left(1+\omega x_{i} q\right) \ldots\left(1+\omega^{t-1} x_{i} q\right) \\
& =\prod_{i=1}^{n}\left(1+\omega^{\frac{t(t-1)}{2}} x_{i}^{t} q^{t}\right)=\sum_{m \geqslant 0} e_{m}\left((-1)^{t-1} X^{t}\right) q^{m t}
\end{aligned}
$$

Comparing coefficients and substituting in (2.2.2) proves (2.2.8). A similar calculation using (2.2.6) and 2.2 .7 proves $(2.2 .9$ and 2.2 .10 respectively.

Remark 2.7. We also consider the monomial and forgotten symmetric functions (see Theorem 3.33, Theorem 3.36 in $t n$ variables specialized at $X, \omega X, \ldots, \omega^{t-1} X$, the same specialization as in Theorem 2.6, in Section 3.3.

### 2.3 Schur and skew Schur polynomials

In this section, we give a brief overview of Schur polynomials and the skew Schur polynomials. Schur polynomials are the characters of irreducible polynomial representations of the general linear group over the field of complex numbers. They also form the most natural basis of the ring of symmetric functions, which are orthonormal with respect to the standard Hall inner product. For a partition $\lambda=\left(\lambda_{1}, \ldots, \lambda_{n}\right)$, the Schur polynomial or general linear character of $\mathrm{GL}_{n}$ is given by the following Weyl character formula:

$$
\begin{equation*}
s_{\lambda}(X)=\frac{\operatorname{det}_{1 \leqslant i, j \leqslant n}\left(x_{i}^{\beta_{j}(\lambda)}\right)}{\operatorname{det}_{1 \leqslant i, j \leqslant n}\left(x_{i}^{n-j}\right)} \tag{2.3.1}
\end{equation*}
$$

The denominator is the standard Vandermonde determinant,

$$
\begin{equation*}
\operatorname{det}_{1 \leqslant i, j \leqslant n}\left(x_{i}^{n-j}\right)=\prod_{1 \leqslant i<j \leqslant n}\left(x_{i}-x_{j}\right) \tag{2.3.2}
\end{equation*}
$$

The following Jacobi-Trudi identity expresses the Schur polynomial $s_{\lambda}(X)$ as a polynomial in the complete symmetric functions,

$$
\begin{equation*}
s_{\lambda}(X)=\operatorname{det}\left(h_{\lambda_{i}-i+j}(X)\right)_{1 \leqslant i, j \leqslant r}, \tag{2.3.3}
\end{equation*}
$$

where $r$ is any integer such that $r \geqslant \ell(\lambda)$. There is also an equivalent formula in terms of the elementary symmetric functions, called the dual Jacobi-Trudi formula,

$$
\begin{equation*}
s_{\lambda}(X)=\operatorname{det}\left(e_{\lambda_{i}^{\prime}-i+j}(X)\right)_{1 \leqslant i, j \leqslant r} \tag{2.3.4}
\end{equation*}
$$

Now we give the combinatorial definition of the Schur polynomials. The combinatorial objects associated with the Schur polynomial $s_{\lambda}(X)$ are semistandard tableaux of shape $\lambda$. Recall that a semistandard tableau or tableau of shape $\lambda$ is a filling of $\lambda$ with entries in $\{1,2, \ldots, n\}$ such that entries increase weekly along rows and strictly along columns. Then the Schur polynomial $s_{\lambda}(X)$ is given by

$$
\begin{equation*}
s_{\lambda}(X)=\sum_{T} \mathrm{wt}(T), \quad \operatorname{wt}(T):=\prod_{i=1}^{n} x_{i}^{c_{i}(T)}, \tag{2.3.5}
\end{equation*}
$$

where the sum is taken over all semistandard tableaux of shape $\lambda$ and $c_{i}(T), i \in[n]$ is the number of occurrences of $i$ in $T$. The formulas (2.3.3), (2.3.4) and (2.3.5) also generalize to infinitely many variables, but (2.3.1) does not.

Example 2.8. Let $n=3$ and consider the partition $\lambda=(2,1)$. Then we have the following tableaux of shape $\lambda$ with entries in $\{1,2,3\}$ :

and thus $s_{(2,1)}\left(x_{1}, x_{2}, x_{3}\right)=x_{0}^{2} x_{1}+x_{0} x_{1}^{2}+x_{0}^{2} x_{2}+2 x_{0} x_{1} x_{2}+x_{1}^{2} x_{2}+x_{0} x_{2}^{2}+x_{1} x_{2}^{2}$.

We note the following relations of the Schur symmetric polynomials with the monomial and forgotten symmetric functions (see [77, Chapter 1, Section 6]). We consider matrices whose rows and columns are indexed by the partitions of $n$. Suppose $K$ and $J$ are two matrices such that $K_{\lambda, \mu}$ is the number of tableaux of shape $\lambda$ and weight $\mu$, and

$$
J_{\lambda, \mu}= \begin{cases}1 & \text { if } \lambda^{\prime}=\mu \\ 0 & \text { otherwise }\end{cases}
$$

Then

$$
\begin{equation*}
m_{\lambda}(X)=\sum_{\mu \varangle \lambda} K_{\lambda, \mu}^{-1} s_{\mu}(X), \tag{2.3.6}
\end{equation*}
$$

$$
\begin{equation*}
f_{\lambda}(X)=\sum_{\mu \dashv|\lambda|}\left(K^{-1} J\right)_{\lambda, \mu} s_{\mu}(X) . \tag{2.3.7}
\end{equation*}
$$

Now we define the skew Schur polynomials, symmetric functions indexed by skew shapes which generalize the Schur polynomials. If $\mu \subset \lambda$ (i.e. $\mu_{i} \leqslant \lambda_{i}, i \geqslant 1$ ), then define the skew Schur function $s_{\lambda / \mu}(X)$ as

$$
\begin{equation*}
s_{\lambda / \mu}(X)=\sum_{T} \mathrm{wt}(T), \quad \operatorname{wt}(T):=\prod_{i \geqslant 1} x_{i}^{c_{i}(T)}, \tag{2.3.8}
\end{equation*}
$$

where the sum is taken over all tableaux of shape $\lambda / \mu$ and $c_{i}(T), i \in[n]$ is the number of occurrences of $i$ in $T$. Otherwise $s_{\lambda / \mu}(X)=0$. The following Jacobi-Trudi formula and dual Jacobi-Trudi formula gives $s_{\lambda / \mu}(X)$ in terms of complete symmetric functions and elementary symmetric functions,

$$
\begin{align*}
& s_{\lambda / \mu}(X)=\operatorname{det}\left(h_{\lambda_{i}-\mu_{j}-i+j}(X)\right)_{1 \leqslant i, j \leqslant r},  \tag{2.3.9}\\
& s_{\lambda / \mu}(X)=\operatorname{det}\left(e_{\lambda_{i}^{\prime}-\mu_{j}^{\prime}-i+j}(X)\right)_{1 \leqslant i, j \leqslant r}, \tag{2.3.10}
\end{align*}
$$

where $r$ is any integer such that $r \geqslant \ell(\lambda)$, which generalize 2.3.3) and 2.3.4 respectively.
The relations (2.3.6) and (2.3.7), and the formulas (2.3.8, 2.3.9) and 2.3.10 also generalize to infinitely many variables, but we do not need them.

### 2.4 Irreducible characters of other classical groups

The characters of irreducible polynomial representations of the symplectic and orthogonal groups are symmetric Laurent polynomials indexed by integer partitions or halfpartitions. These characters are given by the Weyl character formula [40], which describes the characters of irreducible representations of compact Lie groups in terms of their highest weights. In this section, we write down the explicit Weyl character formulas and Jacobi-Trudi-type identities for the characters of classical groups $\mathrm{SO}(2 n+1), \mathrm{Sp}(2 n)$ and $\mathrm{O}(2 n)$ 40].

The odd orthogonal (type B) character of the group $\mathrm{SO}(2 n+1)$ at the representation indexed by the partition $\lambda=\left(\lambda_{1}, \ldots, \lambda_{n}\right)$ is given by

$$
\begin{equation*}
\operatorname{so}_{\lambda}(X)=\frac{\operatorname{det}_{1 \leqslant i, j \leqslant n}\left(x_{i}^{\beta_{j}(\lambda)+1 / 2}-\bar{x}_{i}^{\beta_{j}(\lambda)+1 / 2}\right)}{\operatorname{det}_{1 \leqslant i, j \leqslant n}\left(x_{i}^{n-j+1 / 2}-\bar{x}_{i}^{n-j+1 / 2}\right)}=\frac{\operatorname{det}_{1 \leqslant i, j \leqslant n}\left(x_{i}^{\lambda_{j}+n-j+1}-\bar{x}_{i}^{\lambda_{j}+n-j}\right)}{\operatorname{det}_{1 \leqslant i, j \leqslant n}\left(x_{i}^{n-j+1}-\bar{x}_{i}^{n-j}\right)}, \tag{2.4.1}
\end{equation*}
$$

and the denominator in the first formula is

$$
\begin{equation*}
\operatorname{det}_{1 \leqslant i, j \leqslant n}\left(x_{i}^{n-j+1 / 2}-\bar{x}_{i}^{n-j+1 / 2}\right)=\prod_{i=1}^{n}\left(x_{i}^{1 / 2}-\bar{x}_{i}^{1 / 2}\right) \prod_{1 \leqslant i<j \leqslant n}\left(x_{i}+\bar{x}_{i}-x_{j}-\bar{x}_{j}\right) . \tag{2.4.2}
\end{equation*}
$$

The symplectic (type C) character of the group $\mathrm{Sp}(2 n)$ at the representation indexed by $\lambda=\left(\lambda_{1}, \ldots, \lambda_{n}\right)$ is given by

$$
\begin{equation*}
\operatorname{sp}_{\lambda}(X)=\frac{\operatorname{det}_{1 \leqslant i, j \leqslant n}\left(x_{i}^{\beta_{j}(\lambda)+1}-\bar{x}_{i}^{\beta_{j}(\lambda)+1}\right)}{\operatorname{det}_{1 \leqslant i, j \leqslant n}\left(x_{i}^{n-j+1}-\bar{x}_{i}^{n-j+1}\right)}, \tag{2.4.3}
\end{equation*}
$$

and the denominator here is

$$
\begin{equation*}
\operatorname{det}_{1 \leqslant i, j \leqslant n}\left(x_{i}^{n-j+1}-\bar{x}_{i}^{n-j+1}\right)=\prod_{i=1}^{n}\left(x_{i}-\bar{x}_{i}\right) \prod_{1 \leqslant i<j \leqslant n}\left(x_{i}+\bar{x}_{i}-x_{j}-\bar{x}_{j}\right) . \tag{2.4.4}
\end{equation*}
$$

Lastly, the even orthogonal (type D) character of the group $\mathrm{O}(2 n)$ at the representation indexed by $\lambda=\left(\lambda_{1}, \ldots, \lambda_{n}\right)$ is given by

$$
\begin{equation*}
\mathrm{o}_{\lambda}^{\text {even }}(X)=\frac{2 \operatorname{det}_{1 \leqslant i, j \leqslant n}\left(x_{i}^{\beta_{j}(\lambda)}+\bar{x}_{i}^{\beta_{j}(\lambda)}\right)}{\left(1+\delta_{\lambda_{n}, 0}\right)} \underset{1 \leqslant i, j \leqslant n}{\operatorname{det}}\left(x_{i}^{n-j}+\bar{x}_{i}^{n-j}\right), \tag{2.4.5}
\end{equation*}
$$

where $\delta$ is the Kronecker delta. The extra factor in the denominator arises because of the difference in the representation theory of $\mathrm{O}(2 n)$ and $\mathrm{SO}(2 n)$; see [40, p. 411] and [92, pp. 311-312] for the precise details. The determinant here factorizes as

$$
\begin{equation*}
\operatorname{det}_{1 \leqslant i, j \leqslant n}\left(x_{i}^{n-j}+\bar{x}_{i}^{n-j}\right)=2 \prod_{1 \leqslant i<j \leqslant n}\left(x_{i}+\bar{x}_{i}-x_{j}-\bar{x}_{j}\right) . \tag{2.4.6}
\end{equation*}
$$

Notice that

$$
s_{\lambda}\left(x_{1}, \ldots, x_{n}\right)=\operatorname{sp}_{\lambda}\left(x_{1}, \ldots, x_{n}\right)=\operatorname{so}_{\lambda}\left(x_{1}, \ldots, x_{n}\right)=\mathrm{o}_{\lambda}^{\text {even }}\left(x_{1}, \ldots, x_{n}\right)=0, \quad \text { if } n<\ell(\lambda) .
$$

A half-integer is an odd integer divided by 2 . The expressions (2.4.1) and (2.4.5) for odd and even orthogonal characters, respectively, also hold for half-integer partitions, where a half-integer partition is a tuple $\left(\lambda_{1}, \ldots, \lambda_{n}\right)$ whose entries are all positive halfintegers such that $\lambda_{1} \geqslant \cdots \geqslant \lambda_{n}$. The odd and even orthogonal characters indexed by the half-integer partition $\left(\lambda_{1}+1 / 2, \ldots, \lambda_{n}+1 / 2\right)$ can be expressed in terms of characters
indexed by $\lambda$ as

$$
\begin{gather*}
\operatorname{so}_{\left(\lambda_{1}+1 / 2, \ldots, \lambda_{n}+1 / 2\right)}\left(x_{1}, \ldots, x_{n}\right)=\prod_{i=1}^{n}\left(x_{i}^{1 / 2}+\bar{x}_{i}^{1 / 2}\right) \operatorname{sp}_{\lambda}\left(x_{1}, \ldots, x_{n}\right),  \tag{2.4.7}\\
\mathrm{o}_{\left(\lambda_{1}+1 / 2, \ldots, \lambda_{n}+1 / 2\right)}^{\text {even }}\left(x_{1}, \ldots, x_{n}\right)=(-1)^{\sum_{i=1}^{n} \lambda_{i}} \prod_{i=1}^{n}\left(x_{i}^{1 / 2}+\bar{x}_{i}^{1 / 2}\right) \operatorname{so}_{\lambda}\left(-x_{1}, \ldots,-x_{n}\right) . \tag{2.4.8}
\end{gather*}
$$

We now write the Jacobi-Trudi-type identities for the characters of the other classical groups. The Jacobi-Trudi formula expresses the classical characters as a determinant in terms of the complete homogeneous symmetric polynomials. See [39, 40, 77] for background and more details. The odd orthogonal (type $B$ ) character of the group $\mathrm{SO}(2 n+1)$ is given by

$$
\begin{equation*}
\operatorname{so}_{\lambda}(X)=\operatorname{det}\left(h_{\lambda_{i}-i+j}(X, \bar{X}, 1)-h_{\lambda_{i}-i-j}(X, \bar{X}, 1)\right)_{1 \leqslant i, j \leqslant n} \tag{2.4.9}
\end{equation*}
$$

The symplectic (type $C$ ) character of the group $\operatorname{Sp}(2 n)$ is given by

$$
\begin{equation*}
\operatorname{sp}_{\lambda}(X)=\frac{1}{2} \operatorname{det}\left(h_{\lambda_{i}-i+j}(X, \bar{X})+h_{\lambda_{i}-i-j+2}(X, \bar{X})\right)_{1 \leqslant i \leqslant n} . \tag{2.4.10}
\end{equation*}
$$

Lastly, the even orthogonal (type $D$ ) character of the group $\mathrm{O}(2 n)$ is given by

$$
\begin{equation*}
\mathrm{o}_{\lambda}^{\text {even }}(X)=\operatorname{det}\left(h_{\lambda_{i}-i+j}(X, \bar{X})-h_{\lambda_{i}-i-j}(X, \bar{X})\right)_{1 \leqslant i, j \leqslant n} . \tag{2.4.11}
\end{equation*}
$$

We note that the universal characters of the symplectic and orthogonal groups defined by Koike and Terada [64] are symmetric functions which under the appropriate specializations of the variables become the characters of irreducible polynomial representations of classical groups $\mathrm{SO}(2 n+1), \mathrm{Sp}(2 n)$ and $\mathrm{O}(2 n)$.

### 2.5 The ring of supersymmetric functions

Now we will give a supersymmetric analogue of the symmetric functions defined above. We consider the ring $\mathbb{Z}\left[x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{m}\right]$ of polynomials in $n+m$ independent variables $x_{1}, \ldots, x_{n}, y_{1}, \ldots, y_{m}$ with integer coefficients. Suppose $X=\left(x_{1}, \ldots, x_{n}\right)$ and $Y=\left(y_{1}, \ldots, y_{m}\right)$. A polynomial $f(X, Y)$ in this ring is doubly symmetric if it is separately symmetric in both the $X$ and $Y$ variables. Moreover, if substituting $x_{n}=t$ and $y_{m}=-t$ results in an expression independent of $t$, then we call $f(X, Y)$ a supersymmetric function. Let $\Lambda(X / Y)$ denote the subring of supersymmetric functions. Several bases indexed by partitions are defined on $\Lambda(X / Y)$. The monomial supersymmetric functions
are defined as

$$
M_{\lambda}(X / Y)=\sum_{\mu \cup \nu} m_{\mu}(X) f_{\nu}(Y)
$$

where the sum is over the union $\mu \cup \nu$ of the partitions $\mu$ and $\nu$, which is a partition whose parts are of those of $\mu$ and $\nu$, arranged in descending order. The elementary supersymmetric function indexed by $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}\right)$ is given by

$$
\begin{equation*}
E_{\lambda}(X / Y)=\prod_{i=1}^{n} E_{\lambda_{i}}(X / Y) \tag{2.5.1}
\end{equation*}
$$

where $E_{r}(X / Y)=\sum_{j=0}^{r} e_{j}(X) h_{r-j}(Y), r \geqslant 1$. The complete supersymmetric function indexed by $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}\right)$ is given by

$$
\begin{equation*}
H_{\lambda}(X / Y)=\prod_{i=1}^{n} H_{\lambda_{i}}(X / Y) \tag{2.5.2}
\end{equation*}
$$

where $H_{r}(X / Y)=\sum_{j=0}^{r} h_{j}(X) e_{r-j}(Y), r \geqslant 1$. The power sum supersymmetric polynomial indexed by $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}\right)$ is defined as

$$
\begin{equation*}
P_{\lambda}(X / Y)=\prod_{i=1}^{n} P_{\lambda_{i}}(X / Y), \tag{2.5.3}
\end{equation*}
$$

where $P_{r}(X / Y)=p_{r}(X)+(-1)^{r-1} p_{r}(Y), r \geqslant 1$. We note that $E_{0}(X / Y)=H_{0}(X / Y)=$ $P_{0}(X / Y)=1$. It is convenient to define $E_{r}(X / Y), H_{r}(X / Y)$ and $P_{r}(X / Y)$ to be zero for $r<0$. Now we consider Hook Schur functions (or supersymmetric Schur functions), denoted $\mathrm{hs}_{\lambda}(X / Y)$, supersymmetric functions indexed by integer partitions. They are the characters of irreducible covariant tensor representations of $\mathrm{gl}(\mathrm{m} / n)$ introduced by Berele and Regev [19] in their study of Lie superalgebras. They form a $\mathbb{Z}$-basis of the ring of supersymmetric functions, generalizing Schur polynomials. Additionally, skew hook Schur functions are indexed by the skew shape partitions and generalize skew Schur polynomials. For background, see [57, 79].

Definition 2.9. A supertableau (or semistandard supertableau) $T$ of shape $\lambda / \mu$ with entries

$$
1<2<\cdots<n<1^{\prime}<2^{\prime}<\cdots<m^{\prime}
$$

is a filling of the shape with these entries satisfying the following conditions:

- entries increase weakly along rows and columns
- the unprimed entries strictly increase along rows
- the primed entries strictly increase along columns

We use the shorthand notation $[n] \cup[m]$ to denote the ordered set $\left\{1, \ldots, n, 1^{\prime}, \ldots, m^{\prime}\right\}$ such that $1<\cdots<n<1^{\prime}<\cdots<m^{\prime}$. The weight of a supertableau is given by

$$
\operatorname{wt}(T):=\prod_{i=1}^{n} x_{i}^{n_{i}(T)} \prod_{j=1^{\prime}}^{m^{\prime}} y_{j}^{n_{j}(T)}
$$

where $n_{k}(T), k \in[n] \cup[m]$, is the number of occurrences of $k$ in $T$. Denote the set of supertableaux of skew shape $\lambda / \mu$ with entries in $[n] \cup[m]$ by $\operatorname{SSYT}_{n / m}(\lambda / \mu)$. For integer partitions $\mu \subset \lambda$, the skew hook Schur polynomial, denoted $\mathrm{hs}_{\lambda / \mu}(X / Y)$ is given by:

$$
\begin{equation*}
\mathrm{hs}_{\lambda / \mu}(X / Y):=\sum_{T \in \operatorname{SSYT}_{n / m}(\lambda / \mu)} \mathrm{wt}(T) \tag{2.5.4}
\end{equation*}
$$

We define the skew hook Schur polynomial $\mathrm{hs}_{\lambda / \mu}(X / Y)$ to be zero unless $\mu \subset \lambda$. If $\mu=\varnothing$, then hs $\lambda_{\lambda}(X / Y)$ is the hook Schur function. The hook Schur function hs $\lambda_{\lambda}(X / Y)$ is nonzero if and only if $\lambda_{n+1} \leqslant m$.

Example 2.10. Let $n=2$ and $m=1$ and consider the skew shape $(2,2) /(1)$. Then we have the following supertableaux in $\operatorname{SSYT}_{2 / 1}((2,2) /(1))$ :


|  | 1 |
| :--- | :--- |
| 2 | $1^{\prime}$ |


and thus $\operatorname{hs}_{(2,2) /(1)}\left(x_{1}, x_{2} / y_{1}\right)=x_{1}^{2} x_{2}+x_{1} x_{2}^{2}+x_{1}^{2} y_{1}+2 x_{1} x_{2} y_{1}+x_{2}^{2} y_{1}+x_{1} y_{1}^{2}+x_{2} y_{1}^{2}$. Notice that $\mathrm{hs}_{(2,2) /(1)}\left(x_{1}, t /-t\right)=0$.

We can also define the elements of the ring $\Lambda(X / Y)$ using the notion of plethystic difference $X-Y$. See [22, 46, 72] for background on plethysm and plethystic notation. The skew hook Schur polynomial in plethystic notation is given by

$$
\begin{equation*}
\mathrm{hs}_{\lambda / \mu}(X / Y)=\left.s_{\lambda / \mu}(X-\epsilon Y)\right|_{\epsilon=-1} . \tag{2.5.5}
\end{equation*}
$$

Using the plethystic notation in 2.3.9, we can express $\mathrm{hs}_{\lambda / \mu}(X / Y)$ in terms of the
complete supersymmetric functions.

$$
\begin{equation*}
\operatorname{hs}_{\lambda / \mu}(X / Y)=\operatorname{det}\left(H_{\lambda_{i}-\mu_{j}-i+j}(X / Y)\right)_{1 \leqslant i, j \leqslant r}, \tag{2.5.6}
\end{equation*}
$$

where $r$ is any integer such that $r \geqslant \ell(\lambda)$.

### 2.6 Ribbon tableaux and supertableaux

Recall that a border strip or a ribbon is a connected subdiagram of the Young diagram of $\lambda$ which contains no $2 \times 2$ block of squares. In this section, we define ribbon tableau and ribbon supertableau, generalizing tableau and supertableau respectively.

Definition 2.11. A $k$-horizontal strip is a skew shape formed by a disjoint union of $k$ ribbons such that all their heads are in different columns.

Definition 2.12. A $t$-ribbon tableau (resp. ribbon supertableau) is a filling (resp. supertableau) of shape $\lambda / \mu$ and weight $\nu$ such that the entries along rows and columns are weakly increasing, and the shape determined by the entries labelled $i$, for each $i$, is a $\nu_{i}$-horizontal strip. Such a tableau is called a standard ribbon tableau (resp. standard ribbon supertableau) if the entries are distinct in different ribbons.


Figure 2.1: 4-ribbon tableau of shape $(6,5,5,5,5) /(2,2,1,1)$ and weight $(1,2,0,2)$

Remark 2.13. A 2-ribbon tableau (resp. supertableau) is also known as a domino tableau (resp. supertableau). We denote the set of domino tableaux of shape $\lambda / \mu$ filled with entries in $\{1, \ldots, n\}$ by $\mathcal{D}_{n}(\lambda / \mu)$.

For a ribbon tableau or supertableau $S$, let $\operatorname{Rib}(S)$ denote the set of its ribbons. Recall that the parity is the property of an integer of whether it is even or odd.

Lemma 2.14 ([84, Lemma 4.1], [115, Proposition 3.3.1]). Let $\lambda$ and $\mu$ be partitions such that $\operatorname{core}_{t}(\lambda) / \operatorname{core}_{t}(\mu)$ is empty. Suppose $S$ is a $t$-ribbon tableau of shape $\lambda / \mu$. Then the parity of $\operatorname{ht}(S):=\sum_{\xi \in \operatorname{Rib}(S)} \operatorname{ht}(\xi)$ is independent of $S$.

Definition 2.15. We call a tableau $T$ filled with entries in $\{1,2, \ldots, 2 n\}$ coverable if it can be covered with the dominoes of the form

$a \in\{1,2, \ldots, n\}$. We denote the set of coverable tableaux of shape $\lambda / \mu$ filled with entries in $\{1, \ldots, 2 n\}$ by $\mathcal{C}_{2 n}(\lambda / \mu)$.

The following lemma gives a bijection between the set of coverable tableaux and that of domino tableaux of shape $\lambda / \mu$.

Lemma 2.16. Let $\lambda / \mu$ be a skew shape such that $\operatorname{core}_{2}(\lambda) / \operatorname{core}_{2}(\mu)$ is empty. Then there exists a natural one-to-one correspondence $\psi: \mathcal{C}_{2 n}(\lambda / \mu) \rightarrow \mathcal{D}_{n}(\lambda / \mu)$ such that

$$
d_{i}(\psi(T))=c_{2 i-1}(T)+c_{2 i}(T) \quad \text { and } \quad \sum_{\xi \in \operatorname{Rib}(\psi(T))} \operatorname{ht}(\xi)=c_{1}(T)+\cdots+c_{2 n-1}(T)
$$

where $c_{i}(T)$ and $d_{i}(\psi(T))$ is the number of occurrences of $i$ in $T$ and $\psi(T)$ respectively.
Proof. Suppose $T$ is a coverable tableau. Then define $\psi: \mathcal{C}_{2 n}(\lambda / \mu) \rightarrow \mathcal{D}_{n}(\lambda / \mu)$ such that

$$
\psi(T)(i, j)=\left\lfloor\frac{T(i, j)+1}{2}\right\rfloor
$$

We note that the entries of $\psi(T)$ are weakly increasing along its rows and columns since $T$ is a tableau. Also, $1 \leqslant \psi(T)(i, j) \leqslant n$, for all $(i, j)$. Furthermore, the shape determined by the entries in $\psi(T)$ labelled $i$, for each $i$, is a $\left(\left(c_{2 i-1}(T)+c_{2 i}(T)\right) / 2\right)$-horizontal strip since $T$ is coverable. So, $\psi$ is well-defined and $d_{i}(\psi(T))=c_{2 i-1}(T)+c_{2 i}(T)$. Finally, since $\sum_{\xi \in \operatorname{Rib}(\psi(T))} \mathrm{ht}(\xi)$ is the number of vertical dominoes in $\psi(T)$ and $T$ is coverable, $\sum_{\xi \in \operatorname{Rib}(\psi(T))} \operatorname{ht}(\xi)=c_{1}(T)+\cdots+c_{2 n-1}(T)$.

Example 2.17. Suppose $n=2, \lambda=(5,3,2)$ and $\mu=(1)$. Then the following figure shows a coverable tableau and its image under $\psi$, the bijection defined in Lemma 2.18.

Lemma 2.18. For a skew shape $\lambda / \mu$ and a tableau $T \in \operatorname{SSYT}_{2 n}(\lambda / \mu)$, let

$$
(X,-X)^{T}:=(-1)^{c_{2}(T)+\cdots+c_{2 n}(T)} x_{1}^{c_{1}(T)+c_{2}(T)} x_{2}^{c_{3}(T)+c_{4}(T)} \ldots x_{n}^{c_{2 n-1}(T)+c_{2 n}(T)}
$$

where $c_{j}(T)$ is the number of occurrences on $j$ in $T$ for all $j \in\{1, \ldots, 2 n\}$. Also, suppose


Figure 2.2: A coverable tableau $T$ on the left and the corresponding domino tableau $\psi(T)$ on the right.
$\mathcal{N}_{2 n}(\lambda / \mu)$ is the set of tableaux in $\operatorname{SSYT}_{2 n}(\lambda / \mu)$ which are not coverable. Then

$$
\sum_{T: T \in \mathcal{N}_{2 n}(\lambda / \mu)}(X,-X)^{T}=0 .
$$

Proof. To prove the lemma, it is sufficient to define a fixed-point-free involution $\gamma$ on $\mathcal{N}_{2 n}(\lambda / \mu)$ such that

$$
\begin{equation*}
(X,-X)^{T}=-(X,-X)^{\gamma(T)} . \tag{2.6.1}
\end{equation*}
$$

Let $T \in \mathcal{N}_{2 n}(\lambda / \mu)$. Suppose $i$ is the smallest integer such that $T$ can not be covered with the dominoes of the form


Consider the part of $T$ with entries equal to $2 i-1$ or $2 i$. Some columns of $T$ will have no such entries, while some columns will contain both $2 i-1$ and $2 i$. We ignore these columns. The remaining part will have a certain number $k$ of rows with entries equal to $2 i-1$ or $2 i$ once in each column. Suppose there are $r_{j}$ number of $2 i-1$ followed by a certain number $s_{j}$ of $2 i$ in the $j$ 'th row, for all $1 \leqslant j \leqslant k$. The following diagram shows two such rows:

Since $T$ is not coverable, there will be atleast one $j$ in $\{1, \ldots, k\}$ such that either $s_{j}$ is odd, or $s_{j}$ is even and $r_{j}>0$. Fix the smallest such $j$. In the first case, convert the leftmost $2 i$ to $2 i-1$, and in the second case convert the rightmost $2 i-1$ to $2 i$, in the $j^{\prime}$ 'th row. This will give us $\gamma(T)$. It is easy to see that $\gamma$ is the fixed point-free involution
and it satisfies 2.6.1 , completing the proof.

### 2.7 Cyclic sieving phenomenon

The cyclic sieving phenomenon was introduced by V. Reiner, D. Stanton and D. White 93 generalizing Stembridge's $(-1)$ phenomenon [105, 106, 107]. To define it, let $C_{t}$ be the cyclic group of order $t$ acting on a finite set $X$ and $f(q)$ a polynomial with nonnegative integer coefficients. Then the triple $\left(X, C_{t}, f(q)\right)$ is said to exhibit the cyclic sieving phenomenon (CSP) if, for any integer $k \geqslant 0$,

$$
\left|\left\{x \in X \mid \sigma^{k} \cdot x=x\right\}\right|=f\left(\omega^{k}\right)
$$

where $\sigma$ is a generator of $C_{n}$ and $\omega$ is a primitive $t^{\text {th }}$ root of unity.
At first glance, it might appear odd that evaluating a polynomial with nonnegative integer coefficients at a complex number could result in another nonnegative integer, let alone have any meaningful counting interpretation. However, the cyclic sieving phenomenon (CSP), as extensively explored in the literature, reveals that this occurrence is actually quite common. See [97] for a nice survey on cyclic sieving by Sagan and [1, 5, 24, 43, 62] for various other instances of cyclic sieving. This phenomenon highlights a fascinating interplay between the fields of combinatorics and algebra, with the Reiner-Stanton-White paper [93] serving as a catalyst for a surge in interest in cyclic sieving.

Numerous researchers have explored the cyclic sieving phenomena on the set of semistandard Young tableaux (See [4, 18, 80, 83, 85, 88, 111). Rhoades [94] unveiled a connection between Schützenberger's promotion [102] and the cyclic sieving phenomenon. To state his result, let $\operatorname{SSYT}_{k}(\lambda / \mu)$ of semistandard Young tableaux of shape $\lambda / \mu$ filled with numbers in $\{1, \ldots, k\}$ and $C_{k}$ be the cyclic group of order $k$. Using Kazhdan-Lusztig theory, he showed that if $\lambda$ is rectangular partition of length at most $k$, then the triple

$$
\left(\operatorname{SSYT}_{k}(\lambda), C_{k}, q^{-m(\lambda)} s_{\lambda}\left(1, q, \ldots, q^{k-1}\right)\right),
$$

exhibits the cyclic sieving phenomenon, where $m(\lambda)=\sum_{i=1}^{k}(i-1) \lambda_{i}$ and $s_{\lambda}\left(1, q, \ldots, q^{k-1}\right)$ is the principal specialization of the Schur polynomial. Such a result was further generalized in [38] where cyclic sieving on rectangular $\operatorname{SSYT}_{k}(\lambda)$ with a fixed content vector was considered and also for partition $\lambda$ of any shape with $\operatorname{gcd}(|\lambda|, k)=1[82]$.

Alexandersson, Pfannerer, Rubey and Uhlin proposed the following conjecture [6, Conjecture 50] generalizing Rhoades's result. There exists an action of the cyclic group
$C_{t}$ of order $t$ on $\operatorname{SSYT}_{k}(t \lambda / t \mu)$ such that the triple

$$
\left(\operatorname{SSYT}_{k}(t \lambda / t \mu), C_{t}, s_{t \lambda / t \mu}\left(1, q, \ldots, q^{k-1}\right)\right),
$$

exhibits the cyclic sieving phenomenon. Here $t \lambda / t \mu$ is the stretched Young diagram of $\lambda / \mu$ by $t$. If $t$ does not divide $k$, then the conjecture is false [73]. But the conjecture is true if $k$ is divisible by $t$ [73, Theorem 1.1]. More precisely, it can be reformulated as follows: let $\lambda / \mu$ be a skew partition. If $\lambda_{i}-\mu_{i}$ is divisible by $t$ for all $i \geqslant 1$, then there exists an action of the cyclic group $C_{t}$ of order $t$ such that the triple

$$
\left(\operatorname{SSYT}_{t n}(\lambda / \mu), C_{t}, s_{\lambda / \mu}\left(1, q, \ldots, q^{t n-1}\right)\right)
$$

exhibits the cyclic sieving phenomenon. Recently, in [52], Graeme, Stokke and Wiebe proved the cyclic sieving phenomenon on the set of symplectic tableaux.

## Chapter 3

## Factorization of classical characters twisted by roots of unity

In this chapter, we study the factorization of irreducible characters of representations of $\mathrm{GL}_{t n}, \mathrm{SO}_{2 t n+1}, \mathrm{Sp}_{2 t n}$ and $\mathrm{O}_{2 t n}$, evaluated at elements $\omega^{k} x_{i}$ for $0 \leqslant k \leqslant t-1$ and $1 \leqslant i \leqslant n$. The case of $\mathrm{GL}_{t n}$ was considered by D. J. Littlewood [74] and independently by D. Prasad [90]. In each case, we characterize partitions for which the character value is nonzero in terms of what we call $z$-asymmetric partitions, where $z$ is an integer which depends on the group. We give statements of results and illustrative examples in Section 3.1. We formulate results on beta sets, generating functions and determinant identities in Section 3.2. We give a self-contained proof of Littlewood's result in Section 3.3. In consequence, we also consider the monomial symmetric functions and forgotten symmetric functions with the same specialization in Section 3.3. We prove the new factorizations of other classical characters in Section 3.4. Finally, we prove generating function formulas for $z$-asymmetric partitions and $z$-asymmetric $t$-cores in Section 3.5. This work has appeared in the Journal of Algebra [13].

### 3.1 Main results

The first result in this direction is due to D. Littlewood and independently D. Prasad for $\mathrm{GL}_{t n}$. We will denote our indeterminates by $X, \omega X, \omega^{2} X, \ldots, \omega^{t-1} X$, where we recall that $X=\left(x_{1}, \ldots, x_{n}\right)$ and $\omega$ is a primitive $t^{\prime}$ th root of unity.

For a partition of length at most $t n$, let $\sigma_{\lambda} \in S_{t n}$ be the permutation that rearranges
the parts of $\beta(\lambda)$ such that

$$
\begin{equation*}
\beta_{\sigma_{\lambda}(j)}(\lambda) \equiv q \quad(\bmod t), \quad \sum_{i=0}^{q-1} n_{i}(\lambda)+1 \leqslant j \leqslant \sum_{i=0}^{q} n_{i}(\lambda), \tag{3.1.1}
\end{equation*}
$$

arranged in decreasing order for each $q \in\{0,1, \ldots, t-1\}$. For the empty partition, $\beta(\varnothing, t n)=(t n-1, t n-2, \ldots, 0)$ with $n_{q}(\varnothing, t n)=n, 0 \leqslant q \leqslant t-1$ and

$$
\begin{equation*}
\sigma_{\varnothing}=(t, \ldots, n t, t-1, \ldots, n t-1, \ldots, 1, \ldots,(n-1) t+1) \tag{3.1.2}
\end{equation*}
$$

in one line notation with $\operatorname{sgn}\left(\sigma_{\varnothing}\right)=(-1)^{\frac{t(t-1)}{2} \frac{n(n+1)}{2}}$.
Theorem 3.1 ([74, Equation (7.3;3)], [90, Theorem 2]). Let $\lambda$ be a partition of length at most tn indexing an irreducible representation of $\mathrm{GL}_{t n}$ and $\mathrm{quo}_{t}(\lambda)=\left(\lambda^{(0)}, \ldots, \lambda^{(t-1)}\right)$. Then the $\mathrm{GL}_{t n}$-character $s_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)$ is as follows.

1. If $\operatorname{core}_{t}(\lambda)$ is not empty, then

$$
\begin{equation*}
s_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=0 \tag{3.1.3}
\end{equation*}
$$

2. If $\operatorname{core}_{t}(\lambda)$ is empty, then

$$
\begin{equation*}
s_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=(-1)^{\frac{t(t-1)}{2} \frac{n(n+1)}{2}} \operatorname{sgn}\left(\sigma_{\lambda}\right) \prod_{i=0}^{t-1} s_{\lambda^{(i)}}\left(X^{t}\right) . \tag{3.1.4}
\end{equation*}
$$

In other words, the nonzero $\mathrm{GL}_{t n}$-character is a product of $t \mathrm{GL}_{n}$ characters. We will give a self-contained proof of this result in Section 3.3. We note that Theorem 3.1 for $X=(1)$ is due to Macdonald [77, Chapter I.3, Example 17(a)], where the Schur polynomial on the right hand side of $(\sqrt[3.1 .4]{ })$ is 1 for each $i \in[0, t-1]$. Recently, Karmakar [58] gave a different proof of this result. A similar factorization result is proved by Mizukawa for Schur's $P$ and $Q$ functions [78].

Example 3.2. For $t=2$, Theorem 3.1 says that the character of the group $\mathrm{GL}_{2}$ (i.e., $n=1$ ) of the representation indexed by the partition $(a, b), a \geqslant b \geqslant 0$, evaluated at $(x,-x)$ is nonzero if and only if $a$ and $b$ have the same parity. If $a$ and $b$ are both odd, then

$$
s_{(a, b)}(x,-x)=-s_{\left(\frac{a+1}{2}\right)}\left(x^{2}\right) s_{\left(\frac{b-1}{2}\right)}\left(x^{2}\right),
$$

and if $a$ and $b$ are both even, then

$$
s_{(a, b)}(x,-x)=s_{\left(\frac{b}{2}\right)}\left(x^{2}\right) s_{\left(\frac{a}{2}\right)}\left(x^{2}\right) .
$$

We now generalize Theorem 3.1 to other classical characters. We first need some definitions.

Definition 3.3. Let $z$ be a nonnegative integer. We say that a partition $\lambda$ is $z$ asymmetric if $\lambda=(\alpha \mid \alpha+z)$, in Frobenius coordinates for some strict partition $\alpha$. More precisely, $\lambda=(\alpha \mid \beta)$ where $\beta_{i}=\alpha_{i}+z$ for $1 \leqslant i \leqslant \operatorname{rk}(\lambda)$.

Definition 3.4. A 1-asymmetric partition is said to be symplectid In addition, if a symplectic partition is also a $t$-core, we call it a symplectic $t$-core.

Note that the empty partition is vacuously symplectic. For example, the only symplectic partitions of 6 are $(3,1,1,1)$ and $(2,2,2)$, and the first few symplectic 3 -cores are $(1,1),(2,1,1),(4,2,2,1,1)$ and $(5,3,2,2,1,1)$.

For the symplectic case, we take $G=\mathrm{Sp}_{2 t n}$, the symplectic group of $(2 t n) \times(2 t n)$ matrices. To state our results, it will be convenient to define, for $\lambda=\left(\lambda_{1}, \ldots, \lambda_{k}\right)$, the reverse of $\lambda \operatorname{as~} \operatorname{rev}(\lambda)=\left(\lambda_{k}, \ldots, \lambda_{1}\right)$. Further, if $\mu=\left(\mu_{1}, \ldots, \mu_{j}\right)$ is another partition such that $\mu_{1} \leqslant \lambda_{k}$, then we write the concatenated partition $(\lambda, \mu)=\left(\lambda_{1}, \ldots, \lambda_{k}, \mu_{1}, \ldots, \mu_{j}\right)$.

Theorem 3.5. Let $\lambda$ be a partition of length at most tn indexing an irreducible representation of $\mathrm{Sp}_{2 t n}$ and $\mathrm{quo}_{t}(\lambda)=\left(\lambda^{(0)}, \ldots, \lambda^{(t-1)}\right)$. Then the $\mathrm{Sp}_{2 t n}$-character $\mathrm{sp}_{\lambda}(X, \omega X$, $\left.\ldots, \omega^{t-1} X\right)$ is given as follows.

1. If $\operatorname{core}_{t}(\lambda)$ is not a symplectic $t$-core, then

$$
\begin{equation*}
\operatorname{sp}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=0 \tag{3.1.5}
\end{equation*}
$$

2. If $\operatorname{core}_{t}(\lambda)$ is a symplectic $t$-core with rank $r$, then

$$
\begin{align*}
\operatorname{sp}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=(-1)^{\epsilon} \operatorname{sgn}\left(\sigma_{\lambda}\right) \mathrm{sp}_{\lambda^{(t-1)}}\left(X^{t}\right) & \prod_{i=0}^{\left\lfloor\frac{t-3}{2}\right\rfloor} s_{\mu_{i}^{(1)}}\left(X^{t}, \bar{X}^{t}\right) \\
& \times \begin{cases}\mathrm{so}_{\lambda^{\left(\frac{t}{2}-1\right)}}\left(X^{t}\right) & t \text { even }, \\
1 & t \text { odd },\end{cases} \tag{3.1.6}
\end{align*}
$$

where

$$
\epsilon=-\sum_{i=\left\lfloor\frac{t}{2}\right\rfloor}^{t-2}\binom{n_{i}(\lambda)+1}{2}+ \begin{cases}\frac{n(n+1)}{2}+n r & t \text { even } \\ 0 & t \text { odd }\end{cases}
$$

[^0]and $\mu_{i}^{(1)}=\lambda_{1}^{(t-2-i)}+\left(\lambda^{(i)}, 0, \ldots, 0,-\operatorname{rev}\left(\lambda^{(t-2-i)}\right)\right)$ has $2 n$ parts for $0 \leqslant i \leqslant\left\lfloor\frac{t-3}{2}\right\rfloor$.
Again, nonzero $\mathrm{Sp}_{2 t n}$ characters are a product of characters, but this time there are $\lfloor(t-1) / 2\rfloor \mathrm{GL}_{2 n}$ characters, one $\mathrm{Sp}_{2 n}$ character and, if $t$ is even, one additional $\mathrm{SO}_{2 n+1}$ character. As mentioned above, the only 2 -cores are self-conjugate. Therefore, this character when $t=2$ is nonzero if and only if $\operatorname{core}_{2}(\lambda)=\varnothing$.

Example 3.6. For $t=2$, Theorem 3.5 says that the character of the group $\operatorname{Sp}(4)(n=1)$ of the representation indexed by the partition $(a, b), a \geqslant b \geqslant 0$, evaluated at $(x,-x)$ is nonzero if and only if $a$ and $b$ have the same parity. If $a$ and $b$ are both odd, then

$$
\operatorname{sp}_{(a, b)}(x,-x)=-\operatorname{sp}_{\left(\frac{b-1}{2}\right)}\left(x^{2}\right) \operatorname{so}_{\left(\frac{a+1}{2}\right)}\left(x^{2}\right),
$$

and if $a$ and $b$ are both even, then

$$
\operatorname{sp}_{(a, b)}(x,-x)=\mathrm{sp}_{\left(\frac{a}{2}\right)}\left(x^{2}\right) \mathrm{so}_{\left(\frac{b}{2}\right)}\left(x^{2}\right)
$$

Notice that all the characters on the right-hand side are for the groups $\mathrm{Sp}(2)$ and $\mathrm{SO}(3)$, and in both cases, the partitions indexing them are the 2-quotients and of length 1.

We also give a concrete example.
Example 3.7. Let $n=2, t=3$ and consider the partition $\lambda=(3,2,1,1,1)$ so that $\beta(\lambda)=(8,6,4,3,2,0)$. Hence, $n_{0}(\lambda, 6)=3, n_{1}(\lambda, 6)=1$ and $n_{2}(\lambda, 6)=2$. Hence, it has 3 -core equal to ( 1,1 ), and its symplectic character is nonzero. With $X=\left(x_{1}, x_{2}\right)$, $\mathrm{sp}_{\lambda}\left(X, \omega_{3} X, \omega_{3}^{2} X\right)$ is given by

$$
\left(\frac{\left(x_{1}+x_{2}\right)\left(x_{1} x_{2}+1\right)\left(x_{1}^{2}-x_{2} x_{1}+x_{2}^{2}\right)\left(x_{1}^{2} x_{2}^{2}-x_{1} x_{2}+1\right)}{x_{1}^{3} x_{2}^{3}}\right)^{2} .
$$

Since $\operatorname{quo}_{3}(\lambda)=(\varnothing,(1),(1)), \mu_{0}^{(1)}=0+(1,0,0,0)=(1)$ and we need to calculate $\mathrm{sp}_{(1)}\left(X^{3}\right)$ and $s_{(1)}\left(X^{3}, \bar{X}^{3}\right)$. These are the characters of $\mathrm{Sp}(4), \mathrm{SO}(5)$ respectively, corresponding to the partition (1,0). It turns out that both are equal to

$$
\frac{\left(x_{1}+x_{2}\right)\left(x_{1} x_{2}+1\right)\left(x_{1}^{2}-x_{2} x_{1}+x_{2}^{2}\right)\left(x_{1}^{2} x_{2}^{2}-x_{1} x_{2}+1\right)}{x_{1}^{3} x_{2}^{3}}
$$

verifying Theorem 3.5.
Definition 3.8. A ( -1 )-asymmetric partition is said to be orthogonal. In addition, if an orthogonal partition is also a $t$-core, we call it an orthogonal $t$-core.

Our notion of an orthogonal partition is the same as Macdonald's double of $\alpha$ [77, p. 14], and Garvan-Kim-Stanton's doubled partition of $\alpha$, denoted $\alpha \alpha$ [41, Sec. 8]. The first few orthogonal 3 -cores are $(2),(3,1),(5,3,1,1)$ and $(6,4,2,1,1)$, which are precisely the conjugates of the symplectic 3 -cores listed earlier. Then our result for factorization of even orthogonal characters is as follows.

For the even orthogonal case, we take $G=\mathrm{O}_{2 t n}$, the orthogonal group of $(2 t n) \times(2 t n)$ square matrices.

Theorem 3.9. Let $\lambda$ be a partition of length at most tn indexing an irreducible representation of $\mathrm{O}_{2 \text { tn }}$ and $\mathrm{quo}_{t}(\lambda)=\left(\lambda^{(0)}, \ldots, \lambda^{(t-1)}\right)$. Then the $\mathrm{O}_{2 t n}$ character $\mathrm{o}_{\lambda}^{\text {even }}(X, \omega X$, $\left.\ldots, \omega^{t-1} X\right)$ is given as follows.

1. If $\operatorname{core}_{t}(\lambda)$ is not an orthogonal $t$-core, then

$$
\begin{equation*}
\mathrm{o}_{\lambda}^{\text {even }}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=0 . \tag{3.1.7}
\end{equation*}
$$

2. If $\operatorname{core}_{t}(\lambda)$ is an orthogonal $t$-core with rank $r$, then

$$
\begin{align*}
& \mathrm{o}_{\lambda}^{\text {even }}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=(-1)^{\epsilon} \operatorname{sgn}\left(\sigma_{\lambda}\right) \mathrm{o}_{\lambda}^{e v(0)}\left(X^{t}\right) \prod_{i=1}^{\left\lfloor\frac{t-1}{2}\right\rfloor} s_{\mu_{i}^{(2)}}\left(X^{t}, \bar{X}^{t}\right) \\
& \times \begin{cases}(-1)^{\sum_{i=1}^{n} \lambda_{i}^{(t / 2)}} \operatorname{so}_{\lambda^{(t / 2)}}\left(-X^{t}\right) & t \text { even }, \\
1 & t \text { odd },\end{cases} \tag{3.1.8}
\end{align*}
$$

where

$$
\epsilon=-\sum_{i=\left\lfloor\frac{t+2}{2}\right\rfloor}^{t-1}\binom{n_{i}(\lambda)}{2}+ \begin{cases}\frac{n(n+t-1)}{2}+n r & t \text { even }, \\ \frac{(t-1) n}{2} & t \text { odd },\end{cases}
$$

and $\mu_{i}^{(2)}=\lambda_{1}^{(t-i)}+\left(\lambda^{(i)}, 0, \ldots, 0,-\operatorname{rev}\left(\lambda^{(t-i)}\right)\right)$ has $2 n$ parts for $0 \leqslant i \leqslant\left\lfloor\frac{t-1}{2}\right\rfloor$.
Again, nonzero $\mathrm{O}_{2 \text { tn }}$ characters are a product of characters, but this time there are $\lfloor(t-1) / 2\rfloor \mathrm{GL}_{2 n}$ characters, one $\mathrm{O}_{2 n}$ character and, if $t$ is even, one additional $\mathrm{SO}_{2 n+1}$ character. As in the symplectic factorization in Theorem 3.5, the even orthogonal character for $t=2$ is nonzero if and only if $\operatorname{core}_{2}(\lambda)=\varnothing$. Recall the involution $\check{\omega}$ (see Section 2.2 on the space of symmetric functions the takes $s_{\lambda}$ to $s_{\lambda^{\prime}}$. Koike and Terada have shown [64] that this involution interchanges (universal) orthogonal characters and (universal) symplectic characters. Comparing Theorem 3.5 and Theorem 3.9, it seems reasonable to suppose that we can obtain a proof of the latter from the former using this
involution. However, this involution works only at the level of universal characters and does not commute with our specialization.

Example 3.10. For $t=2$, Theorem 3.9 says that the character of the group $\mathrm{O}(4)$ of the representation indexed by the partition $(a, b), a \geqslant b \geqslant 0$, evaluated at $(x,-x)$ is nonzero if and only if $a$ and $b$ have the same parity. If $a$ and $b$ are both odd, then

$$
\mathrm{o}_{(a, b)}^{\text {even }}(x,-x)=(-1)^{(b+1) / 2} \operatorname{so}_{\left(\frac{b-1}{2}\right)}\left(-x^{2}\right) \mathrm{o}_{\left(\frac{a+1}{2}\right)}^{\text {even }}\left(x^{2}\right),
$$

and if $a$ and $b$ are both even, then

$$
\mathrm{o}_{(a, b)}^{\text {even }}(x,-x)=(-1)^{a / 2} \mathrm{SO}_{\left(\frac{a}{2}\right)}\left(-x^{2}\right) \mathrm{o}_{\left(\frac{b}{2}\right)}^{\text {even }}\left(x^{2}\right) .
$$

Notice that all the characters on the right-hand side are for the groups $\mathrm{SO}(3)$ and $\mathrm{SO}(2)$, and in both cases the partitions indexing them are the 2-quotients and of length 1.

For the odd orthogonal case, we take $G=\mathrm{SO}_{2 \operatorname{tn+1}}$, the orthogonal group of ( $2 t n+$ $1) \times(2 t n+1)$ square matrices. It will turn out that the notion of an 'odd-orthogonal partition' is the same as being self-conjugate, or equivalently, 0 -asymmetric. The first few self-conjugate 3 -cores are (1), $(3,1,1),(4,2,1,1)$ and $(6,4,2,2,1,1)$. Our result for factorization of odd orthogonal characters is as follows.

Theorem 3.11. Let $\lambda$ be a partition of length at most tn indexing an irreducible representation of $\mathrm{SO}_{2 t n+1}$. Then the $\mathrm{SO}_{2 t n+1}$ character $\mathrm{so}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)$ is given as follows.

1. If $\operatorname{core}_{t}(\lambda)$ is not self-conjugate, then

$$
\begin{equation*}
\operatorname{so}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=0 \tag{3.1.9}
\end{equation*}
$$

2. If $\operatorname{core}_{t}(\lambda)$ is self-conjugate with rank $r$, then

$$
\begin{align*}
& \operatorname{so}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=(-1)^{\epsilon} \operatorname{sgn}\left(\sigma_{\lambda}\right) \prod_{i=0}^{\left\lfloor\frac{t-2}{2}\right\rfloor} s_{\mu_{i}^{(3)}}\left(X^{t}, \bar{X}^{t}\right) \\
& \times \begin{cases}\operatorname{so}_{\lambda}\left(\frac{t-1}{2}\right) \\
1 & \left.t X^{t}\right) \\
t \text { odd },\end{cases}  \tag{3.1.10}\\
& t \text { even },
\end{align*}
$$

where

$$
\epsilon=-\sum_{i=\left\lfloor\frac{t+1}{2}\right\rfloor}^{t-1}\binom{n_{i}(\lambda)+1}{2}+ \begin{cases}n r & t \text { odd } \\ 0 & t \text { even }\end{cases}
$$

$$
\text { and } \mu_{i}^{(3)}=\lambda_{1}^{(t-1-i)}+\left(\lambda^{(i)}, 0, \ldots, 0,-\operatorname{rev}\left(\lambda^{(t-1-i)}\right)\right) \text { has } 2 n \text { parts for } 0 \leqslant i \leqslant\left\lfloor\frac{t-2}{2}\right\rfloor \text {. }
$$

Again, nonzero $\mathrm{SO}_{2 t n+1}$ characters are a product of characters, but this time there are $\lfloor t / 2\rfloor \mathrm{GL}_{2 n}$ characters, and, if $t$ is odd, one additional $\mathrm{SO}_{2 n+1}$ character. Since 2-cores are always self-conjugate, odd orthogonal characters always have a nontrivial factorization when $t=2$.

Example 3.12. For $t=2$, Theorem 3.11 says that the character of the group $\mathrm{SO}(5)$ of the representation indexed by the partition $(a, b), a \geqslant b \geqslant 0$, evaluated at $(x,-x)$ is nonzero if and only if $a$ and $b$ have the same parity. We obtain

$$
\mathrm{so}_{(a, b)}(x,-x)=(-1)^{a} s_{\left(\frac{a+b}{2}, 0\right)}\left(x^{2}, \bar{x}^{2}\right)
$$

Notice that the character on the right hand side is for GL(2) and involves the sum of the 2-quotients.

Remark 3.13. It might seem that the results of Theorem 3.5, Theorem 3.9 and Theorem 3.11 are not well-defined because of Remark 2.4. More precisely, the lack of symmetry of the $t$-quotients on the right hand sides of these theorems might cause some worry. However, since changing $n \rightarrow n+1$ will change the length of the partition $\lambda$ by $t n$, the order of the quotients remains unchanged.

Remark 3.14. In some cases, the Schur functions $s_{\mu_{i}^{(j)}}\left(X^{t}, \bar{X}^{t}\right)$ appearing on the right hand sides of Theorem 3.5. Theorem 3.9 and Theorem 3.11 for $j \in[3]$ respectively factorize further into characters of other classical groups, but we do not understand this behavior fully. Whenever $\mu_{i}$ can be written as $\rho_{1}+(\rho,-\operatorname{rev}(\rho))$ or $\rho_{1}+(1+\rho,-\operatorname{rev}(\rho))$ for a partition $\rho$ of length at most $n$, such a factorization occurs by the results in [10]. In that case $s_{\mu_{i}}$ is either a product of two odd orthogonal characters or an even orthogonal and a symplectic character.

Further generalizations of the factorization results have now appeared in [2, 69]. It is natural to ask if there are infinitely many symplectic, orthogonal and self-conjugate $t$-cores. As we have seen, there are no symplectic or orthogonal 2 -cores and all 2 -cores are self-conjugate. For $t \geqslant 3$, it has been proved [41] that there are infinitely many self-conjugate $t$-cores. Our last result gives a generalisation.

Theorem 3.15. There are infinitely many symplectic and orthogonal $t$-cores for $t \geqslant 3$.
This is proved in Section 3.5.

### 3.2 Background results

We collect all the assorted results we will need to prove our main results here. In Section 3.2.1, we will use beta sets of partitions to classify symplectic partitions and their generalizations. We will derive generating functions for such partitions and prove that there are infinitely many of them in Section 3.5. Finally, we will derive determinant identities for block matrices in Section 3.2.2

### 3.2.1 Properties of beta sets

In his treatise, Macdonald [77] used beta sets to derive powerful results for cores and quotients. We review and extend his results to the cases of interest. First, we recall a useful property of the beta numbers. Throughout, we will use the notation $[m]=$ $\{1, \ldots, m\}$ and $\left[m_{1}, m_{2}\right]=\left\{m_{1}, \ldots, m_{2}\right\}$.

Lemma 3.16. Let $\lambda$ and $\mu$ be partitions of length at most $m_{1}$ and $m_{2}$ respectively and let $m_{2} \geqslant \lambda_{1}$. Then $\lambda^{\prime}=\mu$ if and only if the $m_{1}+m_{2}$ numbers $\beta_{j}(\lambda)$ for $j \in\left[m_{1}\right]$ and $m_{1}+m_{2}-1-\beta_{k}(\mu)$ for $k \in\left[m_{2}\right]$ form a permutation of $\left\{0,1, \ldots, m_{1}+m_{2}-1\right\}$.

Proof. The forward implication holds by [77, Chapter I.1, Equation (1.7)].
For the converse, since $m_{2} \geqslant \lambda_{1}$, the $m_{1}+m_{2}$ numbers $\beta_{j}(\lambda)$ for $j \in\left[m_{1}\right]$ and $m_{1}+m_{2}-1-\beta_{k}\left(\lambda^{\prime}\right)$ for $k \in\left[m_{2}\right]$ are a permutation of $\left\{0,1, \ldots, m_{1}+m_{2}-1\right\}$ by [77, Chapter I.1, Equation (1.7)]. So, $\beta_{k}\left(\lambda^{\prime}\right)=\beta_{k}(\mu), k \in\left[m_{2}\right]$ and $\lambda^{\prime}=\mu$.

Let $\lambda, \mu$ be partitions of length at most $m$ such that $\lambda \supset \mu$, and such that the set difference of Young diagrams $\lambda \backslash \mu$ is a border strip of length $t$. Then, it is known [77, Chapter I.1, Example 8(a)] that $\beta(\mu)$ can be obtained from $\beta(\lambda)$ by subtracting $t$ from some part $\beta_{i}(\lambda)$ and rearranging in descending order. Therefore, for a partition $\lambda$ of length at most $m$, we see that

$$
\begin{equation*}
n_{i}(\lambda, m)=n_{i}\left(\operatorname{core}_{t}(\lambda), m\right), \quad 0 \leqslant i \leqslant t-1 . \tag{3.2.1}
\end{equation*}
$$

We now explain the relationship between a partition and its conjugate in terms of their beta sets.

Lemma 3.17. Let $\lambda$ and $\mu$ be partitions of length at most $t m_{1}$ and $t m_{2}$ respectively. If $\mu=\lambda^{\prime}$, then

$$
\begin{equation*}
n_{i}(\lambda)+n_{t-1-i}(\mu)=m_{1}+m_{2}, \quad 0 \leqslant i \leqslant t-1 . \tag{3.2.2}
\end{equation*}
$$

The converse is true if $\lambda$ and $\mu$ are $t$-cores.

Proof. Suppose $\mu=\lambda^{\prime}$. Then Lemma 3.16 implies that the numbers $\beta_{j}(\lambda)$ for $1 \leqslant j \leqslant$ $t m_{1}$ and $t m+t n-1-\beta_{k}(\mu)$ for $1 \leqslant k \leqslant t m_{2}$ are a permutation of $\left\{0,1, \ldots, t m_{1}+t m_{2}-1\right\}$. Since $\xi \equiv t-1-i(\bmod t)$ implies $t m_{1}+t m_{2}-1-\xi \equiv i(\bmod t), n_{i}(\lambda)+n_{t-1-i}(\mu)$ is equal to number of integers in $\left\{0,1, \ldots, t m_{1}+t m_{2}-1\right\}$ congruent to $i(\bmod t)$. Since for each $0 \leqslant i \leqslant t-1$, there are $m_{1}+m_{2}$ numbers in $\left\{0,1, \ldots, t m_{1}+t m_{2}-1\right\}$ congruent to $i$ modulo $t,(3.2 .2$ holds.

Conversely, assume $\lambda$ and $\mu$ are $t$-cores and (3.2.2) holds. Fix $i, 0 \leqslant i \leqslant t-1$. Since $\lambda$ is a $t$-core, the numbers $i<i+t<\cdots<i+\left(n_{i}(\lambda)-1\right) t$ occur in $\beta(\lambda)$. Similarly, since $\mu$ is a $t$-core and $0 \leqslant t-i-1 \leqslant t-1$, the numbers $t-1-i<2 t-1-i<\cdots<\left(n_{t-1-i}(\mu)\right) t-1-i$ occur in $\beta(\mu)$.

So, the parts of $\beta(\lambda)$ and $t m_{1}+t m_{2}-1-\beta(\mu)$ congruent to $i(\bmod t)$ are

$$
i<i+t<\cdots<i+\left(n_{i}(\lambda)-1\right) t
$$

and

$$
t\left(m_{1}+m_{2}-1\right)+i>t\left(m_{1}+m_{2}-2\right)+i>\cdots>\left(n_{i}(\lambda)\right) t+i
$$

respectively. Therefore, all the numbers congruent to $i(\bmod t)$ between $i$ and $t\left(m_{1}+\right.$ $\left.m_{2}-1\right)+i$ appear in the union of $\beta(\lambda)$ and $t m_{1}+t m_{2}-1-\beta(\mu)$. Since this holds for each $0 \leqslant i \leqslant t-1$, parts of $\beta(\lambda)$ and $t m_{1}+t m_{2}-1-\beta(\mu)$ are a permutation of $\left\{0,1, \ldots, t m_{1}+t m_{2}-1\right\}$. Moreover, the largest part of $\beta(\lambda)$ is at most $t\left(m_{1}+m_{2}\right)-1$ implies $\lambda_{1}$ is at most $t m_{2}$. So, by Lemma 3.16, $\mu=\lambda^{\prime}$, completing the proof.

Using Lemma 3.17 and (3.2.1) for $m=t n$, we have the following corollary.

Corollary 3.18. For a partition $\lambda$ of length at most tn, $\operatorname{core}_{t}(\lambda)$ is self-conjugate if and only if

$$
\begin{equation*}
n_{i}(\lambda)+n_{t-1-i}(\lambda)=2 n, \quad 0 \leqslant i \leqslant\left\lfloor\frac{t-1}{2}\right\rfloor . \tag{3.2.3}
\end{equation*}
$$

Recall the definition of $z$-asymmetric partition from Definition 3.3. Let $\mathcal{P}_{z}$ be the set of $z$-asymmetric partitions and $\mathcal{P}_{z, t}$ be the set of $z$-asymmetric $t$-cores.

Lemma 3.19. Let $\lambda=(\alpha \mid \beta)$ be a partition of length at most $m$ and rank $r$. Then the following statements are equivalent.

1. $\lambda \in \mathcal{P}_{z}$.
2. an integer $\xi$ between 0 and $m-z-1$ occurs in $\beta(\lambda)$ if and only if $2 m-z-1-\xi$ does not.
3. $\beta(\lambda)$ is obtained from the sequence $\left(\alpha_{1}+m, \ldots, \alpha_{r}+m, m-1, \ldots, 1,0\right)$ by deleting the numbers $m-z-1-\alpha_{r}>m-z-1-\alpha_{r-1}>\cdots>m-z-1-\alpha_{1}$ lying between 0 and $m$ - 1 .

Proof. First, note that $\lambda \in \mathcal{P}_{z}$ if and only if $\lambda$ is of the form

$$
\lambda=(\alpha_{1}+1, \ldots, \alpha_{r}+r, \underbrace{r, \ldots, r}_{\alpha_{r}+z}, \underbrace{r-1, \ldots, r-1}_{\alpha_{r-1}-\alpha_{r}-1}, \ldots, \underbrace{1, \ldots, 1}_{\alpha_{1}-\alpha_{2}-1}) .
$$

In that case, its beta set is

$$
\begin{aligned}
\beta(\lambda) & =(\alpha_{1}+m, \ldots, \alpha_{r}+m, \underbrace{m-1, \ldots, m-\left(\alpha_{r}+z\right)}_{\alpha_{r}+z}, m-\left(\widehat{\alpha_{r}+z}+1\right), \\
& \underbrace{m-\left(\alpha_{r}+z+2\right), \ldots, m-\left(\alpha_{r-1}+z\right)}_{\alpha_{r-1}-\alpha_{r}-1}, m-\left(\widehat{\alpha_{r-1}+} z+1\right), \ldots, m-\left(\widehat{\alpha_{2}+z}+1\right), \\
& \underbrace{m-\left(\alpha_{2}+z+2\right), \ldots, m-\left(\alpha_{1}+z\right)}_{\alpha_{1}-\alpha_{2}-1}, m-\left(\widehat{\alpha_{1}+z}+1\right), m-\left(\alpha_{1}+z+2\right), \ldots, 0),
\end{aligned}
$$

where a hat on an entry denotes its absence from the tuple. So, Item 1 and Item 3 are equivalent.

Clearly, Item 3 implies Item 2. Now suppose Item 2 holds. Observe that a part of $\beta(\lambda), \lambda_{i}+m-i$ is greater than and equal to $m$ if and only if $\lambda_{i}$ is greater than and equal to $i$. Thus there are $r$ parts of $\beta(\lambda)$ greater than $m$. Since $\alpha_{1}+m>\cdots>\alpha_{r}+m$ are $r$ integers greater than and equal to $m$ which occur in $\beta(\lambda)$, Item 3 holds.

Lemma 3.20. For $2 \leqslant t \leqslant z+1$, the empty partition is the only $t$-core in $\mathcal{P}_{z, t}$.

Proof. Let $\lambda=(\alpha \mid \beta) \in \mathcal{P}_{z}$ have rank $r>0$. Then

$$
\lambda=(\alpha_{1}+1, \ldots, \alpha_{r}+r, \underbrace{r, \ldots, r}_{\alpha_{r}+z}, \underbrace{r-1, \ldots, r-1}_{\alpha_{r-1}-\alpha_{r}-1}, \ldots, \underbrace{1, \ldots, 1}_{\alpha_{1}-\alpha_{2}-1}) .
$$

So, $\lambda_{r+i}=r, 1 \leqslant i \leqslant \alpha_{r}+z$ and $\lambda_{r}^{\prime}=\alpha_{r}+r+z$. If $z \geqslant t-1$, then $\lambda_{r+\alpha_{r}+z-t+1}=r$. Hence, the hook number $h\left(r+\alpha_{r}+z-t+1, r\right)=(r)+\left(\alpha_{r}+r+z\right)-\left(r+\alpha_{r}+z-t+1\right)-(r)+1=t$, which is a contradiction since $\lambda$ is a $t$-core. So, $\lambda$ must be empty.

Now we explain the constraints satisfied by $n_{i}(\lambda), 0 \leqslant i \leqslant t-1$, for a $z$-asymmetric $t$-core $\lambda$ of length at most $t n$.

Lemma 3.21. Let $\lambda$ be a t-core of length at most tn and $0 \leqslant z \leqslant t-2$. Then $\lambda \in \mathcal{P}_{z, t}$ if and only if

$$
\begin{align*}
n_{i}(\lambda)+n_{t-z-1-i}(\lambda) & =2 n \quad \text { for } \quad 0 \leqslant i \leqslant t-z-1,  \tag{3.2.4}\\
\text { and } \quad n_{i}(\lambda) & =n, \quad t-z \leqslant i \leqslant t-1 .
\end{align*}
$$

Proof. Suppose $\lambda=(\alpha \mid \alpha+z)$ and $r k(\lambda)=r$. Using Lemma 3.19(3), $\beta(\lambda)$ is obtained from the sequence $\left(\alpha_{1}+t n, \ldots, \alpha_{r}+t n, t n-1, \ldots, 1,0\right)$ by deleting the numbers $t n-z-1-\alpha_{r}>$ $t n-z-1-\alpha_{r-1}>\cdots>t n-z-1-\alpha_{1}$. Since $n_{i}(\varnothing, t n)=n$ for all $i,(3.2 .4)$ trivially holds for the empty partition. Note that if $t n-z-1-\alpha_{i} \equiv \theta_{i}(\bmod t)$, then $\alpha_{i}+t n \equiv t-z-1-\theta_{i}$ $(\bmod t)$ for $i \in[r]$. In that case $n_{t-z-1-\theta_{i}}(\lambda)$ increases by one and $n_{\theta_{i}}(\lambda)$ decreases by one. Therefore, it is sufficient to show that $\theta_{i} \in[0, t-z-1]$, for each $i \in[r]$ to prove (3.2.4).

We prove this successively in reverse order starting from $\theta_{r}$ and going all the way to $\theta_{1}$. Since $\lambda$ is a $t$-core, if $t n-z-1-\alpha_{r}$ does not occur in $\beta(\lambda)$, then neither does $t n-z-1-\alpha_{r}+t$. Since $t n-z-1-\alpha_{r}$ is the largest number deleted from $(t n-1, t n-2, \ldots, 0)$ to get $\beta(\lambda), t n-z-1-\alpha_{r}+t \geqslant t n$. So, $\alpha_{r}+z+1 \in[z+1, t]$; and $\theta_{r} \in[0, t-z-1]$. There is nothing to show if $\theta_{r-1}=\theta_{r}$. So, assume $\theta_{r-1} \neq \theta_{r}$. Similarly, since $\lambda$ is a $t$-core, if $t n-z-1-\alpha_{r-1}$ does not occur in $\beta(\lambda)$, then neither does $t n-z-1-\alpha_{r-1}+t$. Since $t n-z-1-\alpha_{r-1}$ is the largest number congruent to $\theta_{r-1}$ deleted from $(t n-1, t n-2, \ldots, 0)$ to get $\beta(\lambda), \alpha_{r-1}+z+1 \in[z+1, t]$ and $\theta_{r-1} \in[0, t-z-1]$. Proceeding in this manner, $\theta_{i} \in[0, t-z-1]$ for all $i \in[r]$.

Conversely, assume (3.2.4) holds for $\lambda$. If $\lambda$ is the empty partition, then it belongs to $\mathcal{P}_{z, t}$ vacuously. Now suppose $\lambda$ is non-empty and $\left\{i_{1}, i_{2}, \ldots, i_{k}\right\}_{>} \subset\{0,1, \ldots, t-z-1\}$ such that $n_{i_{j}}(\lambda)>n$ which implies $n_{t-z-1-i_{j}}(\lambda)<n, j \in[k]$. Since $\lambda$ is a $t$-core, for each $j, i_{j}+t n<i_{j}+t(n+1)<\cdots<i_{j}+t\left(n_{i_{j}}(\lambda)-1\right)$ are the parts of $\beta(\lambda)$ greater than and equal to $t n$. If $n_{t-z-1-i_{j}}(\lambda)<n$ for $j \in[k]$ implies parts of $\beta(\lambda)$ less than and equal to $t n-1$ is obtained from the sequence $(\operatorname{tn}-1, t n-2, \ldots, 0)$ by deleting the numbers

$$
t n-z-1-i_{j}, t(n-1)-z-1-i_{j}, \ldots, t\left(n_{t-2-i_{j}}(\lambda)+1\right)-z-1-i_{j} .
$$

Observe that an integer $\xi$ between 0 and $t n-z-1$ occurs in $\beta(\lambda)$ if and only if $2 t n-z-1-\xi$ does not. So, by Lemma 3.19, $\lambda \in \mathcal{P}_{z, t}$.

Corollary 3.22. Let $t \geqslant 3$ and $\lambda$ be a partition of length at most $t n$. Then $\operatorname{core}_{t}(\lambda)$ is a symplectic $t$-core if and only if $n_{i}(\lambda)+n_{t-2-i}(\lambda)=2 n$ for $0 \leqslant i \leqslant\left\lfloor\frac{t-2}{2}\right\rfloor$ and $n_{t-1}(\lambda)=n$. Proof. Set $z=1$ in Lemma 3.21. This now follows using $\ell\left(\operatorname{core}_{t}(\lambda)\right) \leqslant \ell(\lambda) \leqslant t n$ and (3.2.1) for $m=t n$.

Since $\operatorname{core}_{t}(\lambda)^{\prime}=\operatorname{core}_{t}\left(\lambda^{\prime}\right)$ [77, Example I.1(e)], it follows that core $_{t}(\lambda)$ is an orthogonal $t$-core if and only if $\operatorname{core}_{t}\left(\lambda^{\prime}\right)$ is a symplectic $t$-core. We then have the following corollary.

Corollary 3.23. Let $\lambda$ be a partition of length at most tn. Then core $_{t}(\lambda)$ is an orthogonal $t$-core if and only if $n_{0}(\lambda)=n$ and $n_{i}(\lambda)+n_{t-i}(\lambda)=2 n$ for $1 \leqslant i \leqslant\left\lfloor\frac{t}{2}\right\rfloor$.

Proof. Suppose $\ell\left(\lambda^{\prime}\right) \leqslant t m$, for some $m \geqslant 1$. Using Corollary 3.22 for $\lambda^{\prime}$, $\operatorname{core}_{t}(\lambda)$ is an orthogonal $t$-core if and only if $n_{t-1}\left(\lambda^{\prime}\right)=m$ and $n_{i}\left(\lambda^{\prime}\right)+n_{t-2-i}\left(\lambda^{\prime}\right)=2 m$ for $0 \leqslant i \leqslant\left\lfloor\frac{t-2}{2}\right\rfloor$. Now using Lemma 3.17, we get the desired result.

For completeness, we note the following property of the $t$-quotient of orthogonal and symplectic partitions, although we will not use it.

Proposition 3.24 ([41, Bijection 3]). Let $\lambda$ be a partition. If

1. $\lambda^{(0)}$ is an orthogonal partition,
2. $\operatorname{core}_{t}(\lambda)$ is an orthogonal $t$-core, and
3. $\left(\lambda^{(i)}\right)^{\prime}=\lambda^{(t-i)}$ for $\quad 1 \leqslant i \leqslant\left\lfloor\frac{t}{2}\right\rfloor$,
then $\lambda$ is orthogonal. A similar statement holds for symplectic partitions.
We now see how to compute the rank of a $t$-core from its beta-set.
Lemma 3.25. If $\lambda$ is a $t$-core of length at most tn, then

$$
\begin{equation*}
\operatorname{rk}(\lambda)=\sum_{i=0}^{t-1}\left(n_{i}(\lambda)-n\right)_{+}, \tag{3.2.5}
\end{equation*}
$$

where $z_{+}:=\max (z, 0)$.
Proof. If $n_{i}(\lambda)=n$ for $0 \leqslant i \leqslant t-1$, then $\beta(\lambda)=(t n-1, t n-2, \ldots, 1,0)$ which implies $\lambda$ is an empty partition. So, the result holds in this case. Otherwise, assume $\left\{i_{1}, i_{2}, \ldots, i_{k}\right\}_{>} \subset\{0,1, \ldots, t-1\}$ such that $n_{i_{j}}(\lambda)>n$ for $1 \leqslant j \leqslant k$. Since $\lambda$ is a $t$-core,

$$
i_{j}+t n<i_{j}+t(n+1)<\cdots<i_{j}+t\left(n_{i_{j}}(\lambda)-1\right)
$$

are the parts of $\beta(\lambda)$ greater than $t n-1$ for each $j$. If $r$ is the number of parts of $\beta(\lambda)$ greater than $\mathrm{tn}-1$, then

$$
r=\sum_{j=1}^{k}\left(n_{i_{j}}(\lambda)-n\right)=\sum_{i=0}^{t-1}\left(n_{i}(\lambda)-n\right)_{+} .
$$

Moreover, $\beta_{r}(\lambda)$ is the smallest part of $\beta(\lambda)$ greater than $t n-1$ and is therefore equal to $i_{k}+t n$. So, $\lambda_{r}=\beta_{r}(\lambda)-(t n-r)=t n+i_{k}-(t n-r)=i_{k}+r \geqslant r$ and $\lambda_{r+1} \leqslant$ $t n-1-(t n-r-1) \leqslant r$, which implies the rank of $\lambda$ is $r$.

Lemma 3.25immediately tells us how to compute the rank of the $t$-core of a partition using (3.2.1).

Corollary 3.26. If $\lambda$ is a partition of length at most tn, then

$$
\operatorname{rk}\left(\operatorname{core}_{t}(\lambda)\right)=\sum_{i=0}^{t-1}\left(n_{i}(\lambda)-n\right)_{+} .
$$

Lemma 3.25 also gives us an algorithm to determine if a partition has empty $t$-core.

Corollary 3.27. If $\lambda$ is a partition of length at most tn, then core $_{t}(\lambda)$ is empty if and only if $n_{i}(\lambda)=n$ for $0 \leqslant i \leqslant t-1$.

Lemma 3.28. Let $\lambda$ be a partition of length at most tn.

1. If $\operatorname{core}_{t}(\lambda)$ is a symplectic $t$-core, then

$$
\begin{equation*}
\operatorname{rk}\left(\operatorname{core}_{t}(\lambda)\right)=\sum_{i=0}^{\left\lfloor\frac{t-3}{2}\right\rfloor}\left|n_{i}(\lambda)-n\right|=\sum_{i=\left\lfloor\frac{t-1}{2}\right\rfloor}^{t-2}\left|n_{i}(\lambda)-n\right| . \tag{3.2.6}
\end{equation*}
$$

2. If $\operatorname{core}_{t}(\lambda)$ is an orthogonal $t$-core, then

$$
\begin{equation*}
\operatorname{rk}\left(\operatorname{core}_{t}(\lambda)\right)=\sum_{i=1}^{\left\lfloor\frac{t-1}{2}\right\rfloor}\left|n_{i}(\lambda)-n\right| . \tag{3.2.7}
\end{equation*}
$$

3. If $\operatorname{core}_{t}(\lambda)$ is self-conjugate $t$-core, then

$$
\begin{equation*}
\operatorname{rk}\left(\operatorname{core}_{t}(\lambda)\right)=\sum_{i=0}^{\left\lfloor\frac{t-2}{2}\right\rfloor}\left|n_{i}(\lambda)-n\right| . \tag{3.2.8}
\end{equation*}
$$

Proof. Using Corollary 3.26,

$$
\operatorname{rk}\left(\operatorname{core}_{t}(\lambda)\right)=\sum_{i=0}^{t-1}\left(n_{i}(\lambda)-n\right)_{+} .
$$

If core $_{t}(\lambda)$ is a symplectic $t$-core, then by Corollary 3.22,

$$
n_{t-1}(\lambda)=n \text { and } n_{i}(\lambda)+n_{t-2-i}(\lambda)=2 n \text { for } 0 \leqslant i \leqslant\left\lfloor\frac{t-2}{2}\right\rfloor .
$$

If $n_{i}(\lambda)>n$ for some $i \in\left\{\left\lfloor\frac{t-3}{2}\right\rfloor+1,\left\lfloor\frac{t-3}{2}\right\rfloor+2, \ldots, t-2\right\}$, then $n_{t-2-i}(\lambda)<n$ and $n_{i}(\lambda)-n=n-n_{t-2-i}(\lambda)$. Since $t-2-i \in\left\{0,1, \ldots,\left\lfloor\frac{t-3}{2}\right\rfloor\right\}$,

$$
\operatorname{rk}\left(\operatorname{core}_{t}(\lambda)\right)=\sum_{p=0}^{\left\lfloor\frac{t-3}{2}\right\rfloor}\left|n_{p}(\lambda)-n\right| .
$$

Using an argument analogous to the one just given as well as Corollary 3.23 and Corollary 3.18 , the proofs of (3.2.7) and (3.2.8) follow.

### 3.2.2 Determinant evaluations

Here, we will derive all the determinant evaluations we need to prove our character identities. We will state them in the most general form possible.

Let $\lambda$ be a partition with $\ell(\lambda) \leqslant t n$. Recall for $0 \leqslant p \leqslant t-1, \beta_{j}^{(p)}(\lambda), 1 \leqslant j \leqslant n_{p}(\lambda)$ are the parts of $\beta(\lambda)$ congruent to $p$ modulo $t$, arranged in decreasing order. In addition, for $q \in \mathbb{Z} \cup(\mathbb{Z}+1 / 2)$, define $n \times n_{p}(\lambda)$ matrices

$$
\begin{equation*}
A_{p, q}^{\lambda}=\left(x_{i}^{\beta_{j}^{(p)}(\lambda)+q}\right) \underset{1 \leqslant j \leqslant n_{p}(\lambda)}{1 \leqslant}, \quad \bar{A}_{p, q}^{\lambda}=\left(\bar{x}_{i}^{\beta_{j}^{(p)}(\lambda)+q}\right){\underset{c \mid}{1 \leqslant j \leqslant n_{p}(\lambda)}}^{1 \leqslant} . \tag{3.2.9}
\end{equation*}
$$

The corresponding matrices for the empty partition are denoted by

$$
\begin{equation*}
A_{p, q}=\left(x_{i}^{t(n-j)+p+q}\right)_{1 \leqslant i, j \leqslant n}, \quad \bar{A}_{p, q}=\left(\bar{x}_{i}^{t(n-j)+p+q}\right)_{1 \leqslant i, j \leqslant n} \tag{3.2.10}
\end{equation*}
$$

In all cases, whenever $q=0$, we will omit it. For example, we will write $A_{p}^{\lambda}$ instead of $A_{p, 0}^{\lambda}$. Recall that the $t$-quotient of $\lambda$ is given by quo ${ }_{t}(\lambda)=\left(\lambda^{(0)}, \ldots, \lambda^{(t-1)}\right)$ and $n_{p}(\lambda) \leqslant n$ for $0 \leqslant p \leqslant t-1$. Then, using Proposition 2.3(2),

$$
t \beta_{j}\left(\lambda^{(p)}\right)=\beta_{j}^{(p)}(\lambda)-p, \quad 1 \leqslant j \leqslant n
$$

we write down alternate formulas for the classical characters. Recall that $X^{t}=\left(x_{1}^{t}, \ldots, x_{n}^{t}\right)$. Using this notation, we see that the Schur polynomial is given by

$$
\begin{equation*}
s_{\lambda(p)}\left(X^{t}\right)=\frac{\operatorname{det} A_{p}^{\lambda}}{\operatorname{det} A_{p}} \tag{3.2.11}
\end{equation*}
$$

the symplectic character is given by

$$
\begin{equation*}
\operatorname{sp}_{\lambda^{(p)}}\left(X^{t}\right)=\frac{\operatorname{det}\left(A_{p, t-p}^{\lambda}-\bar{A}_{p, t-p}^{\lambda}\right)}{\operatorname{det}\left(A_{p, t-p}-\bar{A}_{p, t-p}\right)} \tag{3.2.12}
\end{equation*}
$$

the odd orthogonal character is given by

$$
\begin{equation*}
\mathrm{so}_{\lambda(p)}\left(X^{t}\right)=\frac{\operatorname{det}\left(A_{p, \frac{t}{2}-p}^{\lambda}-\bar{A}_{p, \frac{t}{2}-p}^{\lambda}\right)}{\operatorname{det}\left(A_{p, \frac{t}{2}-p}-\bar{A}_{p, \frac{t}{2}-p}\right)} \tag{3.2.13}
\end{equation*}
$$

and the even orthogonal character is given by

$$
\begin{equation*}
\mathrm{o}_{\lambda(p)}^{\text {even }}\left(X^{t}\right)=\frac{2 \operatorname{det}\left(A_{p,-p}^{\lambda}+\bar{A}_{p,-p}^{\lambda}\right)}{\left(1+\delta_{\lambda_{n}^{(p)}, 0}\right) \operatorname{det}\left(A_{p,-p}+\bar{A}_{p,-p}\right)}, \tag{3.2.14}
\end{equation*}
$$

using (2.4.3), (2.4.1) and (2.4.5) respectively.
We first express the Schur function in the variables $X^{t} \cup \bar{X}^{t}$ occurring in our theorems.
Lemma 3.29. Let $\lambda$ be a partition of length at most tn with $\operatorname{quo}_{t}(\lambda)=\left(\lambda^{(0)}, \ldots, \lambda^{(t-1)}\right)$. If $p, q \in\{0,1, \ldots, t-1\}$ such that $n_{p}(\lambda)+n_{q}(\lambda)=2 n$, then we define $\rho_{p, q}=\lambda_{1}^{(p)}+$ $\left(\lambda^{(q)}, 0, \ldots, 0,-\operatorname{rev}\left(\lambda^{(p)}\right)\right)$, where we pad $0^{\prime} s$ in the middle so that $\rho_{p, q}$ is of length $2 n$. Then the Schur function $s_{\rho_{p, q}}\left(X^{t}, \bar{X}^{t}\right)$ can be written as

$$
s_{\rho_{p, q}}\left(X^{t}, \bar{X}^{t}\right)=\frac{(-1)^{\frac{n_{p}(\lambda)\left(n_{p}(\lambda)-1\right)}{}}}{(-1)^{\frac{n(n-1)}{2}}} \frac{\operatorname{det}\left(\begin{array}{c|c}
A_{q,-q}^{\lambda} & \bar{A}_{p, t-p}^{\lambda}  \tag{3.2.15}\\
\hline \bar{A}_{q,-q}^{\lambda} & A_{p, t-p}^{\lambda}
\end{array}\right)}{\operatorname{det}\left(\begin{array}{c|c}
A_{q,-q} & \bar{A}_{p, t-p} \\
\hline \bar{A}_{q,-q} & A_{p, t-p}
\end{array}\right) .}
$$

Proof. We will think of the first $n_{q}(\lambda)$ components of $\rho_{p, q}$ as coming from $\lambda^{(q)}$ and the remaining as coming from $\lambda^{(p)}$. Using the Schur polynomial expression (2.3.1), we see that the numerator of $s_{\rho_{p, q}}\left(X^{t}, \bar{X}^{t}\right)$ is

$$
\operatorname{det}\left(\begin{array}{c|c}
\left.\left(x_{i}^{t\left(\lambda_{1}^{(p)}+\lambda_{j}^{(q)}+2 n-j\right)}\right)\right)_{\substack{1 \leqslant i \leqslant n \\
1 \leqslant j \leqslant n_{q}(\lambda)}} & \left(x_{i}^{t\left(\lambda_{1}^{(p)}-\lambda_{2 n+1-j}^{(p)}+2 n-j\right)}\right)_{\substack{1 \leqslant i \leqslant n \\
n_{q}(\lambda)+1 \leqslant j \leqslant 2 n}} \\
\hline\left(\bar{x}_{i}^{t\left(\lambda_{1}^{(p)}+\lambda_{j}^{(q)}+2 n-j\right)}\right)_{\substack{1 \leqslant i \leqslant n \\
1 \leqslant j \leqslant n_{q}(\lambda)}} & \left(\bar{x}_{i}^{t\left(\lambda_{1}^{(p)}-\lambda_{2 n+1-j}^{(p)}+2 n-j\right)}\right)_{\substack{1 \leqslant i \leqslant n \\
n_{q}(\lambda)+1 \leqslant j \leqslant 2 n}}
\end{array}\right) .
$$

Multiplying row $i$ in the top blocks and bottom blocks of the numerator by $\bar{x}_{i}^{t\left(\lambda_{1}^{(p)}+n_{p}(\lambda)\right)}$ and $x_{i}^{t\left(\lambda_{1}^{(p)}+n_{p}(\lambda)\right)}$ respectively, for each $i=1,2, \ldots, n$ and then reversing the last $n_{p}(\lambda)$ columns, we see that the numerator equals

$$
\begin{array}{r}
(-1)^{\frac{n_{p}(\lambda)\left(n_{p}(\lambda)-1\right)}{2}} \operatorname{det}\left(\begin{array}{l|l}
\left(x_{i}^{\beta_{j}^{(q)}(\lambda)-q}\right)_{\substack{1 \leqslant \leqslant n \\
1 \leqslant j \leqslant n_{q}(\lambda)}} \mid\left(\bar{x}_{i}^{\beta_{j}^{(p)}(\lambda)-p+t}\right)_{\substack{1 \leqslant i \leqslant n \\
1 \leqslant j \leqslant n_{p}(\lambda)}} \\
\hline\left(\bar{x}_{i}^{\beta_{j}^{(q)}(\lambda)-q}\right)_{\substack{1 \leqslant i \leqslant n \\
1 \leqslant j \leqslant n_{q}(\lambda)}} & \left(x_{i}^{\beta_{j}^{(p)}(\lambda)-p+t}\right)_{\substack{1 \leqslant i \leqslant n \\
1 \leqslant j \leqslant n_{p}(\lambda)}}
\end{array}\right)  \tag{3.2.16}\\
=(-1)^{\frac{n_{p}(\lambda)\left(n_{p}(\lambda)-1\right)}{2}} \operatorname{det}\left(\begin{array}{c|c}
A_{q,-q}^{\lambda} & \bar{A}_{p, t-p}^{\lambda} \\
\hline \bar{A}_{q,-q}^{\lambda} & A_{p, t-p}^{\lambda}
\end{array}\right) .
\end{array}
$$

Since $\left.\left.n_{p}(\varnothing, t n)\right)=n_{q}(\varnothing, t n)\right)=n$ and the denominator in the expression 2.3.1 is the same as its numerator evaluated at the empty partition, we see that the denominator is

$$
(-1)^{\frac{n(n-1)}{2}} \operatorname{det}\left(\begin{array}{c|c}
A_{q,-q} & \bar{A}_{p, t-p} \\
\hline \bar{A}_{q,-q} & A_{p, t-p}
\end{array}\right) .
$$

Hence, 3.2.15 holds.

The next result shows that the role of $p$ and $q$ in these kind of Schur evaluations can be interchanged.

Lemma 3.30. Using the same notation as in Lemma 3.29, we see that

$$
s_{\rho_{p, q}}(X, \bar{X})=s_{\rho_{q, p}}(X, \bar{X}) .
$$

Proof. Since $\bar{x}_{i}^{t} A_{p, t-p}^{\lambda}=A_{p,-p}^{\lambda}$ and $x_{i}^{t} \bar{A}_{p, t-p}^{\lambda}=\bar{A}_{p,-p}^{\lambda}$, we observe

$$
\left(\begin{array}{c|c}
0 & \bar{x}_{i}^{t} I_{n} \\
\hline x_{i}^{t} I_{n} & 0
\end{array}\right)\left(\begin{array}{c|c}
A_{q,-q}^{\lambda} & \bar{A}_{p, t-p}^{\lambda} \\
\hline \bar{A}_{q,-q}^{\lambda} & A_{p, t-p}^{\lambda}
\end{array}\right)\left(\begin{array}{c|c}
0 & I_{n_{q}(\lambda)} \\
\hline I_{n_{p}(\lambda)} & 0
\end{array}\right)=\left(\begin{array}{c|c}
A_{p,-p}^{\lambda} & \bar{A}_{q, t-q}^{\lambda} \\
\hline \bar{A}_{p,-p}^{\lambda} & A_{q, t-q}^{\lambda}
\end{array}\right),
$$

where $I_{m}$ is the $m \times m$ identity matrix. Evaluating the determinant on both sides,

$$
(-1)^{n^{2}} \operatorname{det}\left(\begin{array}{c|c}
A_{q,-q}^{\lambda} & \bar{A}_{p, t-p}^{\lambda} \\
\hline \bar{A}_{q,-q}^{\lambda} & A_{p, t-p}^{\lambda}
\end{array}\right)(-1)^{n_{p}(\lambda) n_{q}(\lambda)}=\operatorname{det}\left(\begin{array}{c|c}
A_{p,-p}^{\lambda} & \bar{A}_{q, t-q}^{\lambda} \\
\hline \bar{A}_{p,-p}^{\lambda} & A_{q, t-q}^{\lambda}
\end{array}\right) .
$$

Since

$$
n^{2}+\frac{n_{p}(\lambda)\left(n_{p}(\lambda)-1\right)}{2}+n_{p}(\lambda) n_{q}(\lambda)+\frac{n_{q}(\lambda)\left(n_{q}(\lambda)-1\right)}{2}=n^{2}+2 n^{2}-n=n(n-1)
$$

is even, the sign cancels, and $p$ and $q$ can be interchanged.
The remaining results in this section deal with determinants of block matrices, which will prove useful in evaluating the other classical characters. We note that we have not found our identities in Krattenthaler's treatises [67, 68].

Lemma 3.31. For $i=1, \ldots, k$, let $T_{i}$ be matrices of order $\ell_{i} \times m_{i}$ such that $\ell_{1}+\cdots+\ell_{k}=$ $m_{1}+\cdots+m_{k}=d$. Define block-diagonal and block-antidiagonal matrices

$$
U:=\left(\begin{array}{cccc}
T_{1} & & & \\
& T_{2} & & 0 \\
& & \ddots & \\
0 & & & T_{k}
\end{array}\right) \quad \text { and } \quad V:=\left(\begin{array}{cccc} 
& & & T_{1} \\
0 & & T_{2} & \\
& . & & \\
T_{k} & & & 0
\end{array}\right) .
$$

Then

$$
\operatorname{det}(U)=(-1)^{\Sigma_{1 \leqslant i<j \leqslant k} m_{i} m_{j}} \operatorname{det}(V)= \begin{cases}0 & \text { if } \ell_{i} \neq m_{i} \text { for some } i, \\ \prod_{i=1}^{k} \operatorname{det}\left(T_{i}\right) & \text { otherwise } .\end{cases}
$$

Proof. It is easy to see that if $\ell_{i}=m_{i}$ for all $i$, then

$$
\operatorname{det}(U)=\prod_{i=1}^{k} \operatorname{det}\left(T_{i}\right), \quad \operatorname{det}(V)=(-1)^{\Sigma_{1 \leqslant i<j \leqslant k} m_{i} m_{j}} \operatorname{det}(U)
$$

Now, assume $T_{i}$ is not a square matrix for some $i \in[k]$. We use $M^{t}$ to denote the transpose of a matrix $M$. Suppose first that $\ell_{i}<m_{i}$. Then, since $\operatorname{rank}\left(T_{i}^{t} T_{i}\right) \leqslant$ $\operatorname{rank}\left(T_{i}\right) \leqslant \ell_{i}<\operatorname{order}\left(T_{i}^{t} T_{i}\right), \operatorname{det}\left(T_{i}^{t} T_{i}\right)=0$. Therefore,

$$
(\operatorname{det} U)^{2}=\operatorname{det} U^{t} U=\prod_{j=1}^{k} \operatorname{det}\left(T_{j}^{t} T_{j}\right)=0,
$$

which implies $\operatorname{det}(U)=0$, and thus $\operatorname{det}(V)=0$. If $\ell_{i}>m_{i}$, a similar calculation using the rank of $T_{i} T_{i}^{t}$ yields the same result.

Lemma 3.32. Suppose $u_{1}, \ldots, u_{k}$ are positive integers summing up to $k n$. Further, let $\left(\gamma_{i, j}\right)_{1 \leqslant i \leqslant k, 1 \leqslant j \leqslant k+1}$ be a matrix of parameters such that $\gamma_{i, k+1}=\gamma_{i, k}, 1 \leqslant i \leqslant k$ and $\Gamma$ be the square matrix consisting of its first $k$ columns. Let $U_{j}$ and $V_{j}$ be matrices of order $n \times u_{j}$ for $j \in[k]$. Finally, define a $k n \times k n$ matrix with $k \times k$ blocks as

$$
\Pi:=\left(\left.\begin{array}{ll}
\left(\gamma_{i, 2 j-1} U_{j}-\gamma_{i, 2 j} V_{j}\right) & \begin{array}{c}
1 \leqslant i \leqslant k \\
1 \leqslant j \leqslant\left\lfloor\frac{k+1}{2}\right\rfloor
\end{array}
\end{array} \right\rvert\,\left(\gamma_{i, 2 k+2-2 j} U_{j}-\gamma_{i, 2 k+1-2 j} V_{j}\right) \underset{\substack{1 \leqslant i \leqslant k \\
\left\lfloor\frac{k+3}{2}\right\rfloor \leqslant j \leqslant k}}{ }\right) .
$$

1. If $u_{p}+u_{k+1-p} \neq 2 n$ for some $p \in[k]$, then $\operatorname{det} \Pi=0$.
2. If $u_{p}+u_{k+1-p}=2 n$ for all $p \in[k]$, then

$$
\begin{equation*}
\operatorname{det} \Pi=(-1)^{\Sigma}(\operatorname{det} \Gamma)^{n} \prod_{i=1}^{\left\lfloor\frac{k+1}{2}\right\rfloor} \operatorname{det} W_{i}, \tag{3.2.17}
\end{equation*}
$$

where

$$
W_{i}= \begin{cases}\left(\begin{array}{c|c}
U_{i} & -V_{k+1-i} \\
\hline-V_{i} & U_{k+1-i}
\end{array}\right) & 1 \leqslant i \leqslant\left\lfloor\frac{k}{2}\right\rfloor, \\
\left(U_{\frac{k+1}{2}}-V_{\frac{k+1}{2}}\right) & k \text { odd and } i=\frac{k+1}{2},\end{cases}
$$

and

$$
\Sigma=\sum_{i=1}^{\left\lfloor\frac{k}{2}\right\rfloor}\left(n+u_{i}\right)+ \begin{cases}0 & k \text { even } \\ n \sum_{i=1}^{\frac{k-1}{2}} u_{i} & k \text { odd } .\end{cases}
$$

Proof. Consider the permutation $\zeta$ in $S_{k n}$ which rearranges the columns of $\Pi$ blockwise in the following order: $1, k, 2, k-1, \ldots$. In other words, $\zeta$ can be written in one line notation as

$$
\begin{aligned}
& \zeta=(\underbrace{1, \ldots, u_{1}}_{u_{1}}, \underbrace{k n-u_{k}+1, \ldots, k n}_{u_{k}}, \\
&\underbrace{u_{1}+1, \ldots, u_{1}+u_{2}}_{u_{2}}, \underbrace{k n-u_{k}-u_{k-1}+1, \ldots, k n-u_{k}}_{u_{k-1}}, \ldots) .
\end{aligned}
$$

Then, the number of inversions of $\zeta$ is

$$
\begin{align*}
\operatorname{inv}(\zeta)= & \sum_{i=\left\lfloor\frac{k+3}{2}\right\rfloor}^{k} u_{i}\left(k n-\left(u_{1}+\cdots+u_{k+1-i}\right)-\left(u_{i}+\cdots+u_{k}\right)\right) \\
& =\sum_{i=1}^{\left\lfloor\frac{k}{2}\right\rfloor} u_{k+1-i}\left(k n-\left(u_{1}+\cdots+u_{k+1-i}\right)-\left(u_{i}+\cdots+u_{k}\right)\right) . \tag{3.2.18}
\end{align*}
$$

Then it can be seen that

$$
\begin{equation*}
\operatorname{det} \Pi=\operatorname{sgn}(\zeta) \operatorname{det}\left(\gamma_{i, j} U_{j^{\prime \prime}}-\gamma_{i, j^{\prime}} V_{j^{\prime \prime}}\right)_{1 \leqslant i, j \leqslant k}, \tag{3.2.19}
\end{equation*}
$$

where

$$
j^{\prime}=j-(-1)^{j} \quad \text { and } \quad j^{\prime \prime}= \begin{cases}\frac{j+1}{2} & j \text { odd } \\ k+1-\frac{j}{2} & j \text { even }\end{cases}
$$

Now note that

$$
\left(\gamma_{i, j} U_{j^{\prime \prime}}-\gamma_{i, j^{\prime}} V_{j^{\prime \prime}}\right)_{1 \leqslant i, j \leqslant k}=\left(\gamma_{i, j} I_{n}\right)_{1 \leqslant i, j \leqslant k} \times\left(\begin{array}{cccc}
W_{1} & & & \\
& W_{2} & & 0 \\
& & \ddots & \\
0 & & & W_{\left\lfloor\frac{k+1}{2}\right\rfloor}
\end{array}\right)
$$

Now, the matrix $\left(\gamma_{i, j} I_{n}\right)_{1 \leqslant i, j \leqslant k}$ can be written as a tensor product $\Gamma \otimes I_{n}$ and therefore $\operatorname{det}\left(\gamma_{i, j} I_{n}\right)_{1 \leqslant i, j \leqslant k}=(\operatorname{det} \Gamma)^{n}$. Therefore,

$$
\operatorname{det}\left(\gamma_{i, j} U_{j^{\prime \prime}}-\gamma_{i, j^{\prime}} V_{j^{\prime \prime}}\right)_{1 \leqslant i, j \leqslant k}=(\operatorname{det} \Gamma)^{n} \operatorname{det}\left(\begin{array}{cccc}
W_{1} & & &  \tag{3.2.20}\\
& W_{2} & & 0 \\
& & \ddots & \\
0 & & & W_{\left\lfloor\frac{k+1}{2}\right\rfloor}
\end{array}\right)
$$

If $u_{p}+u_{k+1-p} \neq 2 n$, for some $p \in\left[\left\lfloor\frac{k+1}{2}\right\rfloor\right]$, then $W_{p}$ is not a square matrix. Using Lemma 3.31, we see that the latter determinant is zero, Hence, by (3.2.20) and (3.2.19),

$$
\operatorname{det} \Pi=0 .
$$

Now suppose $u_{p}+u_{k+1-p}=2 n, \forall p \in\left[\left\lfloor\frac{k+1}{2}\right\rfloor\right]$. Then $W_{p}$ is a square matrix $\forall p \in\left[\left\lfloor\frac{k+1}{2}\right\rfloor\right]$.

So, by (3.2.20), we get,

$$
\operatorname{det}\left(\gamma_{i, j} U_{j^{\prime \prime}}-\gamma_{i, j^{\prime}} V_{j^{\prime \prime}}\right)_{1 \leqslant i, j \leqslant k}=(\operatorname{det} \Gamma)^{n} \prod_{i=1}^{\left\lfloor\frac{k+1}{2}\right\rfloor} \operatorname{det} W_{i} .
$$

All that remains is to compute the sign. By (3.2.18), we get

$$
\operatorname{inv}(\zeta)=\sum_{i=1}^{\left\lfloor\frac{k}{2}\right\rfloor}\left(2 n-u_{i}\right)(k n-2(k-i-1))
$$

Therefore, if $k$ is even, then $\operatorname{inv}(\zeta)$ is even and $\operatorname{sgn}(\zeta)$ is 1 . If $k$ is odd, then the only contribution for $\operatorname{sgn}(\zeta)$ comes from $n \sum_{i=1}^{\frac{k-1}{2}} u_{i}$, since other terms are even. Summing the terms gives $\Sigma$, completing the proof.

### 3.3 Schur factorization

We first give a self-contained proof of Theorem 3.1, the result of Littlewood [74]. We note that Littlewood's strategy of proof is, although in a different language, essentially the same as ours. Next, we consider the monomial symmetric functions with the specialization $X, \omega X, \ldots, \omega^{t-1} X$, where we recall that $X=\left(x_{1}, \ldots, x_{n}\right)$ and $\omega$ is a primitive $t^{\text {th }}$ root of unity. Then we use the result to count the number of terms in the expansion of $\Theta(\mathbb{Z} / t \mathbb{Z})^{n}$, where $\Theta(\mathbb{Z} / t \mathbb{Z})$ is the group determinant of the cyclic group $\mathbb{Z} / t \mathbb{Z}$. Finally, we consider forgotten symmetric functions with the same specialization.

Proof of Theorem 3.1. Recall that $\lambda$ has length at most $t n$. From the definition 2.3.1, the desired Schur polynomial is

$$
\begin{equation*}
s_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=\frac{\operatorname{det}\left(\left(\left(\omega^{p-1} x_{i}\right)^{\beta_{j}(\lambda)}\right)_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant t n}}\right)_{1 \leqslant p \leqslant t}}{\operatorname{det}\left(\left(\left(\omega^{p-1} x_{i}\right)^{t n-j}\right)_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant t n}}\right)_{1 \leqslant p \leqslant t}} . \tag{3.3.1}
\end{equation*}
$$

Permuting the columns of the determinant in the numerator of (3.3.1) by $\sigma_{\lambda}$ from (3.1.1),
we see that the numerator of (3.3.1) is

$$
\begin{gather*}
\operatorname{sgn}\left(\sigma_{\lambda}\right) \operatorname{det}\left(\left(\left(\omega^{p-1} x_{i}\right)^{\beta_{\lambda}(j)(\lambda)}\right)_{\substack{1 \leqslant l \leqslant n \\
1 \leqslant j \leqslant t n}}\right)_{1 \leqslant p \leqslant t} \\
=\operatorname{sgn}\left(\sigma_{\lambda}\right) \operatorname{det}\left(\omega^{(p-1)(q-1)}\left(x_{i}^{\beta_{j}^{(q-1)}(\lambda)}\right)_{\substack{1 \leqslant i \leqslant n \\
1 \leqslant n_{q-1}(\lambda)}}\right)_{1 \leqslant p, q \leqslant t}  \tag{3.3.2}\\
=\operatorname{sgn}\left(\sigma_{\lambda}\right) \operatorname{det}\left(\omega^{(p-1)(q-1)} A_{q-1}^{\lambda}\right)_{1 \leqslant p, q \leqslant t},
\end{gather*}
$$

where $A_{q-1}^{\lambda}$ is defined in (3.2.9). Note that

$$
\left(\omega^{(p-1)(q-1)} A_{q-1}^{\lambda}\right)_{1 \leqslant p, q \leqslant t}=\left(\omega^{(p-1)(q-1)} \otimes I_{n}\right)_{1 \leqslant p, q \leqslant t} \times\left(\begin{array}{cccc}
A_{0}^{\lambda} & & & \\
& A_{1}^{\lambda} & & 0 \\
0 & & \ddots & \\
& & & A_{t-1}^{\lambda}
\end{array}\right)
$$

where $I_{n}$ is the $n \times n$ identity matrix. Hence,

$$
\begin{align*}
& \operatorname{det}\left(\omega^{(p-1)(q-1)} A_{q-1}^{\lambda}\right)_{1 \leqslant p, q \leqslant t} \\
& \quad=\left(\operatorname{det}\left(\omega^{(p-1)(q-1)}\right)_{1 \leqslant p, q \leqslant t}\right)^{n} \times \operatorname{det}\left(\begin{array}{cccc}
A_{0}^{\lambda} & & & \\
& A_{1}^{\lambda} & & 0 \\
0 & & \ddots & \\
& & & A_{t-1}^{\lambda}
\end{array}\right) . \tag{3.3.3}
\end{align*}
$$

If $\operatorname{core}_{t}(\lambda)$ is not empty, then using Corollary 3.27, we see that $n_{q}(\lambda) \neq n$ for some $0 \leqslant q \leqslant t-1$. So, $A_{q}^{\lambda}$ is not a square matrix for some $0 \leqslant q \leqslant t-1$. By Lemma 3.31, $\operatorname{det}\left(\omega^{(p-1)(q-1)} A_{q-1}^{\lambda}\right)_{1 \leqslant p, q \leqslant t}=0$ and hence

$$
s_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=0
$$

If $\operatorname{core}_{t}(\lambda)$ is empty, then Corollary 3.27 shows that $n_{q}(\lambda)=n$ for all $0 \leqslant q \leqslant t-1$ and $A_{q}^{\lambda}$ is a square matrix for all $0 \leqslant q \leqslant t-1$. Applying Lemma 3.31 again to (3.3.3), we see that

$$
\operatorname{det}\left(\omega^{(p-1)(q-1)} A_{q-1}^{\lambda}\right)_{1 \leqslant p, q \leqslant t}=\left(\operatorname{det}\left(\omega^{(p-1)(q-1)}\right)_{1 \leqslant p, q \leqslant t}\right)^{n} \prod_{q=0}^{t-1} \operatorname{det} A_{q}^{\lambda} .
$$

Substituting in (3.3.2), we see that the numerator of (3.3.1) is

$$
\begin{equation*}
\operatorname{sgn}\left(\sigma_{\lambda}\right)\left(\operatorname{det}\left(\omega^{(p-1)(q-1)}\right)_{1 \leqslant p, q \leqslant t}\right)^{n} \prod_{q=0}^{t-1} \operatorname{det} A_{q}^{\lambda} . \tag{3.3.4}
\end{equation*}
$$

Evaluating (3.3.4) for the empty partition and using (3.1.2), we see that the denominator of (3.3.1) is

$$
\begin{equation*}
(-1)^{\frac{t(t-1)}{2} \frac{n(n+1)}{2}}\left(\operatorname{det}\left(\omega^{(p-1)(q-1)}\right)_{1 \leqslant p, q \leqslant t}\right)^{n} \prod_{q=0}^{t-1} \operatorname{det} A_{q} . \tag{3.3.5}
\end{equation*}
$$

Substitution of the values (3.3.4) and (3.3.5) in (3.3.1) gives

$$
\begin{equation*}
s_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=(-1)^{\frac{t(t-1)}{2} \frac{n(n+1)}{2}} \operatorname{sgn}\left(\sigma_{\lambda}\right) \prod_{q=0}^{t-1} \frac{\operatorname{det} A_{q}^{\lambda}}{\operatorname{det} A_{q}}, \tag{3.3.6}
\end{equation*}
$$

where $A_{q}$ is defined in (3.2.10). Hence, using (3.2.11) in (3.3.6) gives

$$
s_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=(-1)^{\frac{t(t-1)}{2} \frac{n(n+1)}{2}} \operatorname{sgn}\left(\sigma_{\lambda}\right) \prod_{i=0}^{t-1} s_{\lambda^{(i)}}\left(X^{t}\right),
$$

completing the proof.
Now we consider the specialized monomial symmetric function evaluated at the elements $X, \omega X, \ldots, \omega^{t-1} X$, where we recall that $X=\left(x_{1}, \ldots, x_{n}\right)$ and $\omega$ is a primitive $t^{\text {th }}$ root of unity.

Theorem 3.33. Let $\lambda$ be a partition of length at most tn. Then the monomial symmetric function $m_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)$ is given by

1. If $|\lambda| \not \equiv 0(\bmod t)$, then

$$
\begin{equation*}
m_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=0 . \tag{3.3.7}
\end{equation*}
$$

2. If $|\lambda| \equiv 0(\bmod t)$, then

$$
\begin{align*}
& m_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right) \\
& \quad=(-1)^{\frac{n(n+1)}{2} \frac{t(t-1)}{2}}\left(\sum_{\begin{array}{l}
\mu \triangleleft \lambda, \\
\operatorname{coret}_{t}(\mu)=\varnothing
\end{array}} \operatorname{sgn}\left(\sigma_{\mu}\right) K_{\lambda, \mu}^{-1} s_{\mu^{(0)}}\left(X^{t}\right) \ldots s_{\mu^{(t-1)}}\left(X^{t}\right)\right), \tag{3.3.8}
\end{align*}
$$

where $K_{\lambda, \mu}$ is the number of tableaux of shape $\lambda$ and weight $\mu$

Proof of Theorem 3.33. Assume $|\lambda| \not \equiv 0(\bmod t)$. If $\mu$ is a partition of size $|\lambda|$, then core $_{t}(\mu)$ is non-empty and $s_{\mu}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=0$. Recall, by (2.3.6), the desired monomial symmetric function is

$$
\begin{equation*}
m_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=\sum_{\mu \Delta \lambda} K_{\lambda, \mu}^{-1} s_{\mu}\left(X, \omega X, \ldots, \omega^{t-1} X\right), \tag{3.3.9}
\end{equation*}
$$

where $K_{\lambda, \mu}$ is the number of tableaux of shape $\lambda$ and weight $\mu$. This implies

$$
m_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=0
$$

If $|\lambda| \equiv 0(\bmod t)$, then using Theorem 3.1 in (3.3.9) completes the proof.
Theorem 3.34. Let $\lambda$ be a partition of length at most tn. Then $m_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=$ 0 if and only if $|\lambda| \not \equiv 0(\bmod t)$.

Proof. Assume $m_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=0$. Then by (2.3.6), we have

$$
\begin{equation*}
\sum_{\mu \triangleleft \lambda} K_{\lambda, \mu}^{-1} s_{\mu}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=0 \tag{3.3.10}
\end{equation*}
$$

Since $K_{\lambda, \mu}>0$ for all $\mu \vDash \lambda$, we have

$$
s_{\mu}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=0, \text { for all } \mu \& \lambda
$$

So, by Theorem 3.1, $\operatorname{core}_{t}(\mu)$ is non-empty for all $\mu \leqslant \lambda$. Since $\underbrace{(1, \ldots, 1)}_{|\lambda|}$ is the smallest partition in the dominance partial order, $\operatorname{core}_{t}(\underbrace{(1, \ldots, 1)}_{|\lambda|})$ is non-empty. Therefore, $|\lambda| \not \equiv$ $0(\bmod t)$. Hence $m_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=0$ implies $|\lambda| \not \equiv 0(\bmod t)$. The converse follows from Theorem 3.33.

Remark 3.35. For a finite group $G=\left\{g_{1}, \ldots, g_{n}\right\}$, the group determinant $\Theta(G)$ of $G$ is defined as follows

$$
\Theta(G):=\sum_{\sigma \in S_{n}} \operatorname{sgn}(\sigma) x_{g_{1} g_{\sigma(1)}^{-1}} x_{g_{2} g_{\sigma(2)}^{-1}} \ldots x_{g_{n} g_{\sigma(n)}^{-1}} .
$$

Recently in [121], the authors gave an expression ([121, Theorem 3.2]) for $\Theta(\mathbb{Z} / t \mathbb{Z})^{n}$, where the coefficient of $x_{1}^{\lambda_{1}} \ldots x_{n}^{\lambda_{n}}$ is $m_{\lambda}\left(1, \omega, \ldots, \omega^{t n-1}\right)$. Therefore by Theorem 3.34, the number of terms in the expansion of $\Theta(\mathbb{Z} / t \mathbb{Z})^{n}$ is the same as the number of partitions $\lambda$ of length $t n$ such that $|\lambda| \equiv 0(\bmod t)$ and $\lambda_{i} \leqslant t$ for all $i$.

A similar calculation as in the proof of Theorem 3.33 using (2.3.7) proves the following result for the specialized forgotten symmetric functions.

Theorem 3.36. Let $\lambda$ be a partition of length at most tn. Then the forgotten symmetric function $f_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)$ is given by

1. If $|\lambda| \not \equiv 0(\bmod t)$, then

$$
\begin{equation*}
f_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=0 \tag{3.3.11}
\end{equation*}
$$

2. If $|\lambda| \equiv 0(\bmod t)$, then

$$
\begin{align*}
& f_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right) \\
& \quad=(-1)^{\frac{n(n+1)}{2} \frac{t(t-1)}{2}}\left(\sum_{\begin{array}{c}
\mu \vdash|\lambda|, \\
\operatorname{coret}^{\prime}(\mu)=\varnothing
\end{array}} \operatorname{sgn}\left(\sigma_{\mu}\right)\left(K^{-1} J\right)_{\lambda, \mu} s_{\mu^{(0)}}\left(X^{t}\right) \ldots s_{\mu^{(t-1)}}\left(X^{t}\right)\right), \tag{3.3.12}
\end{align*}
$$

where $K_{\lambda, \mu}$ is the number of tableaux of shape $\lambda$ and weight $\mu$

### 3.4 Factorization of other classical characters

In this section, we will prove all the other classical character factorizations using results from Section 3.2. We will give the most details for the symplectic case in Section 3.4.1 and will be a little more sketchy for the even orthogonal case in Section 3.4.2 and the odd orthogonal case in Section 3.4.3. We will assume $\ell(\lambda) \leqslant t n$ throughout this section.

### 3.4.1 Symplectic characters

We first recall the matrices $A_{p, q}^{\lambda}$ and $\bar{A}_{p, q}^{\lambda}$ from (3.2.9). If $\sum_{i=0}^{t-2} n_{i}(\lambda)=(t-1) n$, then consider the $(t-1) n \times(t-1) n$ matrix

$$
\Pi_{1}=\left(\omega^{p q} A_{q-1,1}^{\lambda}-\bar{\omega}^{p q} \bar{A}_{q-1,1}^{\lambda}\right)_{1 \leqslant p, q \leqslant t-1} .
$$

Substitution of $U_{j}=A_{j-1,1}^{\lambda}, V_{j}=\bar{A}_{j-1,1}^{\lambda}$ for $1 \leqslant j \leqslant t-1$ and

$$
\gamma_{i, j}= \begin{cases}\omega^{\frac{i(j+1)}{2}} & j \text { odd }, \\ \omega^{-\frac{i j}{2}} & j \text { even },\end{cases}
$$

in Lemma 3.32 proves the following corollary.
Corollary 3.37. 1. If $n_{i}(\lambda)+n_{t-2-i}(\lambda) \neq 2 n$ for some $i \in\left[0,\left\lfloor\frac{t-2}{2}\right\rfloor\right]$, then $\operatorname{det} \Pi_{1}=0$.
2. If $n_{i}(\lambda)+n_{t-2-i}(\lambda)=2 n$ for all $i \in\left\{0,1, \ldots,\left\lfloor\frac{t-2}{2}\right\rfloor\right\}$, then

$$
\begin{align*}
\operatorname{det} \Pi_{1}=(-1)^{\Sigma_{1}}\left(\operatorname{det}\left(\gamma_{i, j}\right)_{1 \leqslant i, j \leqslant t-1}\right)^{n} & \prod_{q=1}^{\left\lfloor\frac{t-1}{2}\right\rfloor}
\end{align*} \operatorname{det}\left(\begin{array}{l|l}
A_{q-1,1} & \bar{A}_{t-q-1,1} \\
\hline \bar{A}_{q-1,1} & A_{t-q-1,1}
\end{array}\right) \quad \begin{array}{ll}
\operatorname{det}\left(A_{\frac{t}{2}-1,1}-\bar{A}_{\frac{t}{2}-1,1}\right) & \text { t even }  \tag{3.4.1}\\
1 & t \text { odd }
\end{array}
$$

where

$$
\Sigma_{1}=\sum_{q=1}^{\left\lfloor\frac{t-1}{2}\right\rfloor}\left(n+n_{q-1}(\lambda)\right)+ \begin{cases}n \sum_{q=1}^{\frac{t-2}{2}} n_{q-1}(\lambda) & \text { t even } \\ 0 & t \text { odd }\end{cases}
$$

Proof of Theorem 3.5. Using the formula for symplectic characters in (2.4.3), we see that the symplectic polynomial considered here is

$$
\begin{equation*}
\operatorname{sp}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=\frac{\operatorname{det}\left(\left(\left(\omega^{p} x_{i}\right)^{\beta_{j}(\lambda)+1}-\left(\bar{\omega}^{p} \bar{x}_{i}\right)^{\beta_{j}(\lambda)+1}\right)_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant n}}\right)_{0 \leqslant p \leqslant t-1}}{\operatorname{det}\left(\left(\left(\omega^{p} x_{i}\right)^{t n-j+1}-\left(\bar{\omega}^{p} \bar{x}_{i}\right)^{t n-j+1}\right)_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant t n}}\right)_{0 \leqslant p \leqslant t-1}} \tag{3.4.2}
\end{equation*}
$$

Since the denominator of the right hand side of (3.4.2) is the same as its numerator evaluated at the empty partition, we compute the factorization for the numerator and use that to get factorization for the denominator. Permuting the columns of the determinant in the numerator of (3.4.2) by $\sigma_{\lambda}$ from (3.1.1), we see that the numerator of (3.4.2) is

$$
\begin{gather*}
\operatorname{sgn}\left(\sigma_{\lambda}\right) \operatorname{det}\left(\left(\left(\omega^{p} x_{i}\right)^{\beta_{\sigma_{\lambda}(j)}(\lambda)+1}-\left(\bar{\omega}^{p} \bar{x}_{i}\right)^{\beta_{\sigma_{\lambda}(j)}(\lambda)+1}\right)_{\substack{1 \leqslant i \leqslant n \\
1 \leqslant j \leqslant t n}}\right)_{0 \leqslant p \leqslant t-1} \\
\left.=\operatorname{sgn}\left(\sigma_{\lambda}\right) \operatorname{det}\left(\left(\omega^{p(q+1)} x_{i}^{\beta_{j}^{(q)}(\lambda)+1}-\bar{\omega}^{p(q+1)} \bar{x}_{i}^{\beta_{j}^{(q)}(\lambda)+1}\right)\right)_{\substack{1 \leqslant i \leqslant n \\
1 \leqslant j \leqslant n_{q}(\lambda)}}\right)_{0 \leqslant p, q \leqslant t-1}  \tag{3.4.3}\\
=\operatorname{sgn}\left(\sigma_{\lambda}\right) \operatorname{det}\left(\omega^{p(q+1)} A_{q, 1}^{\lambda}-\bar{\omega}^{p(q+1)} \bar{A}_{q, 1}^{\lambda}\right)_{0 \leqslant p, q \leqslant t-1} .
\end{gather*}
$$

Applying the blockwise row operations $R_{1} \rightarrow R_{1}+R_{2}+\cdots+R_{t}$ followed by $R_{i} \rightarrow R_{i}-\frac{1}{t} R_{1}$,
for $2 \leqslant i \leqslant t$, we get

$$
\begin{align*}
& \operatorname{det}\left(\omega^{p(q+1)} A_{q, 1}^{\lambda}-\bar{\omega}^{p(q+1)} \bar{A}_{q, 1}^{\lambda}\right)_{0 \leqslant p, q \leqslant t-1} \\
& \left.\qquad \begin{array}{c|c|c|c}
0 & \ldots & 0 & t\left(A_{t-1,1}^{\lambda}-\bar{A}_{t-1,1}^{\lambda}\right) \\
\omega A_{0,1}^{\lambda}-\omega^{t-1} \bar{A}_{0,1}^{\lambda} & \ldots & \omega^{t-1} A_{t-2,1}^{\lambda}-\omega \bar{A}_{t-2,1}^{\lambda} & 0 \\
\hline & \\
\hline \operatorname{det}^{2} A_{0,1}^{\lambda}-\omega^{t-2} \bar{A}_{0,1}^{\lambda} & \ldots & \omega^{t-2} A_{t-2,1}^{\lambda}-\omega^{2} \bar{A}_{t-2,1}^{\lambda} & 0 \\
\hline \vdots & \ddots & & \\
\hline \omega^{t-1} A_{0,1}^{\lambda}-\omega \bar{A}_{0,1}^{\lambda} & \ldots & \omega A_{t-2,1}^{\lambda}-\omega^{t-1} \bar{A}_{t-2,1}^{\lambda} & 0
\end{array}\right) . \tag{3.4.4}
\end{align*}
$$

This is now a $2 \times 2$ block determinant with anti-diagonal blocks. We apply Lemma 3.31. for $k=2$ and $d=t n$, to evaluate this determinant.

If $\operatorname{core}_{t}(\lambda)$ is not a symplectic $t$-core, then by Corollary 3.22, either $n_{t-1}(\lambda) \neq n$ or $n_{i}(\lambda)+n_{t-2-i}(\lambda) \neq 2 n$ for some $i \in\left\{0,1, \ldots,\left\lfloor\frac{t-2}{2}\right\rfloor\right\}$. In the first case, i.e. if $n_{t-1}(\lambda) \neq$ $n$, then Lemma 3.31 shows the determinant is (3.4.4) is 0 . If $n_{t-1}(\lambda)=n$, then the determinant in (3.4.4) is

$$
\begin{equation*}
(-1)^{(t-1) n^{2}} t^{n} \operatorname{det}\left(A_{t-1,1}^{\lambda}-\bar{A}_{t-1,1}^{\lambda}\right) \times \operatorname{det}\left(\omega^{p q} A_{q-1,1}^{\lambda}-\bar{\omega}^{p q} \bar{A}_{q-1,1}^{\lambda}\right)_{1 \leqslant p, q \leqslant t-1}, \tag{3.4.5}
\end{equation*}
$$

using Lemma 3.31. Observe that the $(t-1) n \times(t-1) n$ block matrix of the determinant in (3.4.5) is of the form $\Pi_{1}$ in Corollary 3.37. Now if $n_{i}(\lambda)+n_{t-2-i}(\lambda) \neq 2 n$ for some $i \in\left\{0,1, \ldots,\left\lfloor\frac{t-2}{2}\right\rfloor\right\}$, then the determinant in (3.4.5) is 0 by Lemma 3.32 and therefore, in both cases,

$$
\operatorname{sp}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=0
$$

If $\operatorname{core}_{t}(\lambda)$ is a symplectic $t$-core, then by Corollary 3.22, $n_{t-1}(\lambda)=n$ and $n_{i}(\lambda)+$ $n_{t-2-i}(\lambda)=2 n, i \in\left\{0,1, \ldots,\left\lfloor\frac{t-2}{2}\right\rfloor\right\}$. Using Corollary 3.37(2) in (3.4.5), we see that the
determinant in the numerator of $(3.4 .2)$ is

$$
\begin{align*}
& \operatorname{sgn}\left(\sigma_{\lambda}\right)\left((-1)^{(t-1) n} t\right)^{n} \operatorname{det}\left(A_{t-1,1}^{\lambda}-\bar{A}_{t-1,1}^{\lambda}\right)(-1)^{\Sigma_{1}}\left(\operatorname{det}\left(\gamma_{i, j}\right)_{1 \leqslant i, j \leqslant t-1}\right)^{n} \\
& \quad \times \prod_{q=1}^{\left\lfloor\frac{t-1}{2}\right\rfloor} \operatorname{det}\left(\begin{array}{ll}
A_{q-1,1}^{\lambda} & \bar{A}_{t-q-1,1}^{\lambda} \\
\hline \bar{A}_{q-1,1}^{\lambda} & A_{t-q-1,1}^{\lambda}
\end{array}\right) \times \begin{cases}\operatorname{det}\left(A_{\frac{t}{2}-1,1}^{\lambda}-\bar{A}_{\frac{t}{2}-1,1}^{\lambda}\right) & t \text { even }, \\
1 & t \text { odd } .\end{cases} \tag{3.4.6}
\end{align*}
$$

We now simplify the $2 \times 2$ block determinants. For $1 \leqslant q \leqslant\left\lfloor\frac{t-1}{2}\right\rfloor$, multiplying row $i$ in the top blocks of the matrix by $\bar{x}_{i}^{q}$ and row $i$ in the bottom blocks by $x_{i}^{q}$ for each $i \in[n]$, we get

$$
\begin{align*}
& \operatorname{det}\left(\begin{array}{l|l}
A_{q-1,1}^{\lambda} & \bar{A}_{t-q-1,1}^{\lambda} \\
\hline \bar{A}_{q-1,1}^{\lambda} & A_{t-q-1,1}^{\lambda}
\end{array}\right) \\
= & \operatorname{det}\left(\begin{array}{l|l}
\left(x_{i}^{\beta_{j}^{(q-1)}(\lambda)+1-q}\right)_{\substack{1 \leqslant i \leqslant n \\
1 \leqslant j \leqslant n_{q-1}(\lambda)}} & \left(\bar{x}_{i}^{\beta_{j}^{(t-1-q)}(\lambda)+1+q}\right)_{\substack{1 \leqslant j i \leqslant n \\
1 \leqslant j \leqslant n_{t-1}-q(\lambda)}} \\
\hline\left(\bar{x}_{i}^{\beta_{j}^{(q-1)}(\lambda)+1-q}\right)_{\substack{1 \leqslant i \leqslant n \\
1 \leqslant j n_{q-1}(\lambda)}} & \left(x_{i}^{\beta_{j}^{(t-1-q)}(\lambda)+1+q}\right)_{\substack{1 \leqslant i \leqslant n \\
1 \leqslant j \leqslant n_{t-1-q}(\lambda)}}
\end{array}\right)  \tag{3.4.7}\\
= & \operatorname{det}\left(\begin{array}{c|c}
A_{q-1,1-q}^{\lambda} & \bar{A}_{t-q-1, q+1}^{\lambda} \\
\hline \bar{A}_{q-1,1-q}^{\lambda} & A_{t-q-1, q+1}^{\lambda}
\end{array}\right) .
\end{align*}
$$

Combining (3.4.6) and (3.4.7), we see that the numerator of (3.4.2) is given by

$$
\begin{align*}
& \operatorname{sgn}\left(\sigma_{\lambda}\right)\left((-1)^{(t-1) n} t\right)^{n} \operatorname{det}\left(A_{t-1,1}^{\lambda}-\bar{A}_{t-1,1}^{\lambda}\right)(-1)^{\Sigma_{1}}\left(\operatorname{det}\left(\gamma_{i, j}\right)_{1 \leqslant i, j \leqslant t-1}\right)^{n} \\
& \quad \times \prod_{q=1}^{\left\lfloor\frac{t-1}{2}\right\rfloor} \operatorname{det}\left(\begin{array}{ll}
A_{q-1,1-q}^{\lambda} & \bar{A}_{t-q-1, q+1}^{\lambda} \\
\hline \bar{A}_{q-1,1-q}^{\lambda} & A_{t-q-1, q+1}^{\lambda}
\end{array}\right) \times \begin{cases}\operatorname{det}\left(A_{\frac{t}{2}-1,1}^{\lambda}-\bar{A}_{\frac{t}{2}-1,1}^{\lambda}\right) & t \text { even }, \\
1 & t \text { odd. }\end{cases} \tag{3.4.8}
\end{align*}
$$

Evaluating $(\sqrt{3.4 .8})$ at the empty partition and using $(3.1 .2)$, we see that the denominator of (3.4.2) is given by

$$
\begin{align*}
&(-1)^{\frac{t(t-1)}{2} \frac{n(n+1)}{2}}\left((-1)^{(t-1) n} t\right)^{n} \operatorname{det}\left(A_{t-1,1}-\bar{A}_{t-1,1}\right)(-1)^{\Sigma_{1}^{0}}\left(\operatorname{det}\left(\gamma_{i, j}\right)_{1 \leqslant i, j \leqslant t-1}\right)^{n} \\
& \times \prod_{q=1}^{\left\lfloor\frac{t-1}{2}\right\rfloor} \operatorname{det}\left(\begin{array}{ll}
A_{q-1,1-q} & \bar{A}_{t-q-1, q+1} \\
\hline \bar{A}_{q-1,1-q} & A_{t-q-1, q+1}
\end{array}\right) \times \begin{cases}\operatorname{det}\left(A_{\frac{t}{2}-1,1}-\bar{A}_{\frac{t}{2}-1,1}\right) & t \text { even } \\
1 & t \text { odd }\end{cases} \tag{3.4.9}
\end{align*}
$$

where

$$
\Sigma_{1}^{0}=\sum_{q=1}^{\left\lfloor\frac{t-1}{2}\right\rfloor} 2 n+ \begin{cases}0 & t \text { odd } \\ n \sum_{q=1}^{\frac{t-2}{2}} n & t \text { even }\end{cases}
$$

For $0 \leqslant i \leqslant\left\lfloor\frac{t-3}{2}\right\rfloor$, let $\mu_{i}^{(1)}=\lambda_{1}^{(t-2-i)}+\left(\lambda^{(i)}, 0, \ldots, 0,-\operatorname{rev}\left(\lambda^{(t-2-i)}\right)\right)$. Since $n_{i}(\lambda)+$ $n_{t-2-i}(\lambda)=2 n$, Lemma 3.29 gives

$$
s_{\mu_{i}^{(1)}}\left(X^{t}, \bar{X}^{t}\right)=\frac{\left.(-1)^{\left(n_{t-i-2}(\lambda)\right.}\right)}{(-1)^{\binom{n}{2}}} \frac{\operatorname{det}\left(\begin{array}{c|c}
A_{i,-i}^{\lambda} & \bar{A}_{t-2-i, i+2}^{\lambda}  \tag{3.4.10}\\
\hline \bar{A}_{i,-i}^{\lambda} & A_{t-2-i, i+2}^{\lambda}
\end{array}\right)}{\operatorname{det}\left(\begin{array}{c|c}
A_{i,-i} & \bar{A}_{t-2-i, i+2} \\
\hline \bar{A}_{i,-i} & A_{t-2-i, i+2}
\end{array}\right) .}
$$

Now substitute (3.4.8) and (3.4.9) in (3.4.2), and then use (3.2.12) for $p=t-1$, (3.2.13) for $p=\frac{t}{2}-1$ and 3.4 .10 for $0 \leqslant i \leqslant\left\lfloor\frac{t-3}{2}\right\rfloor$. The symplectic character is thus given by

$$
\begin{aligned}
& \operatorname{sp}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right) \\
& \quad=(-1)^{\epsilon} \operatorname{sgn}\left(\sigma_{\lambda}\right) \operatorname{sp}_{\lambda^{(t-1)}}\left(X^{t}\right) \prod_{i=0}^{\left\lfloor\frac{t-3}{2}\right\rfloor} s_{\mu_{i}^{(1)}}\left(X^{t}, \bar{X}^{t}\right) \times \begin{cases}\text { so }_{\lambda^{\left(\frac{t}{2}-1\right)}}\left(X^{t}\right) & t \text { even }, \\
1 & t \text { odd },\end{cases}
\end{aligned}
$$

where

$$
\begin{aligned}
\epsilon=\frac{t(t-1)}{2} \frac{n(n+1)}{2}+\sum_{q=0}^{\left\lfloor\frac{t-3}{2}\right\rfloor}\left(n_{q}(\lambda)\right. & -n) \times \begin{cases}n+1 & t \text { even } \\
1 & t \text { odd }\end{cases} \\
& +\sum_{i=0}^{\left\lfloor\frac{t-3}{2}\right\rfloor}\left(\frac{n(n-1)}{2}-\frac{n_{t-i-2}(\lambda)\left(n_{t-i-2}(\lambda)-1\right)}{2}\right) .
\end{aligned}
$$

It remains to compute the sign by simplifying the expression for $\epsilon$. Since for $0 \leqslant q \leqslant$ $\left\lfloor\frac{t-3}{2}\right\rfloor, n_{q}(\lambda)+n_{t-2-q}(\lambda)=2 n$, replacing $n_{q}(\lambda)-n$ by $n-n_{t-2-q}(\lambda)$ in the first summation and then using the facts that $\frac{(t-1)(t+1)}{2} \frac{n(n+1)}{2}$ is even for odd $t$ and the parity of $\frac{n(n+1)}{2} \frac{\left(t^{2}-2\right)}{2}$ is the same as the parity of $\frac{n(n+1)}{2}$ for odd $t$ shows that $\epsilon$ has the same parity as

$$
\begin{aligned}
&-\sum_{i=0}^{\left\lfloor\frac{t-3}{2}\right\rfloor}\binom{n_{t-2-i}(\lambda)+1}{2}+ \begin{cases}\frac{n(n+1)}{2} & +n r \\
0 & t \text { even }\end{cases} \\
&=-\sum_{i=\left\lfloor\frac{t}{2}\right\rfloor}^{t-2}\binom{n_{i}(\lambda)+1}{2}+ \begin{cases}\frac{n(n+1)}{2}+n r & t \text { even } \\
0 & t \text { odd }\end{cases}
\end{aligned}
$$

where $r$ is the rank from Lemma 3.28(1). This completes the proof.

### 3.4.2 Even orthogonal characters

We first recall the matrices $A_{p, q}^{\lambda}$ and $\bar{A}_{p, q}^{\lambda}$ from (3.2.9). If $\sum_{i=1}^{t-1} n_{i}(\lambda)=(t-1) n$, then consider the $(t-1) n \times(t-1) n$ block matrix

$$
\Pi_{2}=\left(\omega^{p q} A_{q}^{\lambda}+\bar{\omega}^{p q} \bar{A}_{q}^{\lambda}\right)_{1 \leqslant p, q \leqslant t-1}
$$

Substitution of $U_{j}=A_{j}^{\lambda}, V_{j}=-\bar{A}_{j}^{\lambda}$ and for $1 \leqslant j \leqslant t-1$,

$$
\gamma_{i, j}= \begin{cases}\omega^{\frac{i(j+1)}{2}} & j \text { odd } \\ \omega^{-\frac{i j}{2}} & j \text { even }\end{cases}
$$

in Lemma 3.32 proves the following corollary.

Corollary 3.38. 1. If $n_{i}(\lambda)+n_{t-i}(\lambda) \neq 2 n$ for some $i \in\left[\left\lfloor\frac{t}{2}\right\rfloor\right]$, then $\operatorname{det} \Pi_{2}=0$.
2. If $n_{i}(\lambda)+n_{t-i}(\lambda)=2 n$ for all $i \in\left[\left\lfloor\frac{t}{2}\right\rfloor\right]$, then

$$
\begin{align*}
\operatorname{det} \Pi_{2}=(-1)^{\Sigma_{2}}\left(\operatorname{det}\left(\gamma_{i, j}\right)_{1 \leqslant i, j \leqslant t-1}\right)^{n} & \prod_{q=1}^{\left\lfloor\frac{t-1}{2}\right\rfloor} \operatorname{det}\left(\begin{array}{l|l}
A_{q}^{\lambda} & \bar{A}_{t-q}^{\lambda} \\
\hline \bar{A}_{q}^{\lambda} & A_{t-q}^{\lambda}
\end{array}\right)  \tag{3.4.11}\\
& \times \begin{cases}\operatorname{det}\left(A_{\frac{t}{2}}^{\lambda}+\bar{A}_{\frac{t}{2}}^{\lambda}\right) & t \text { even } \\
1 & t \text { odd }\end{cases}
\end{align*}
$$

where

$$
\Sigma_{2}= \begin{cases}n \sum_{q=1}^{\frac{t-2}{2}} n_{q}(\lambda) & t \text { even } \\ 0 & t \text { odd }\end{cases}
$$

We now give a sketch of the proof of Theorem 3.9 following similar ideas as in the proof of Theorem 3.5.

Proof of Theorem 3.9. Using the formula for even orthogonal characters is 2.4.5), we see that desired polynomial is

$$
\begin{equation*}
\mathrm{o}_{\lambda}^{\text {even }}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=\frac{2 \operatorname{det}\left(\left(\left(\omega^{p-1} x_{i}\right)^{\beta_{j}(\lambda)}+\left(\bar{\omega}^{p-1} \bar{x}_{i}\right)^{\beta_{j}(\lambda)}\right)_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant t n}}\right)_{1 \leqslant p \leqslant t}}{\left(1+\delta_{\lambda t n, 0}\right) \operatorname{det}\left(\left(\left(\omega^{p-1} x_{i}\right)^{t n-j}+\left(\bar{\omega}^{p-1} \bar{x}_{i}\right)^{t n-j}\right)_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant t n}}\right)_{\substack{1 \leqslant p \leqslant t}} .} \tag{3.4.12}
\end{equation*}
$$

After permuting the columns of the determinant in the numerator of (3.4.12) by $\sigma_{\lambda}$ from (3.1.1), we see that the numerator of (3.4.12) becomes

$$
\begin{equation*}
2 \operatorname{sgn}\left(\sigma_{\lambda}\right) \operatorname{det}\left(\omega^{(p-1)(q-1)} A_{q-1}^{\lambda}+\bar{\omega}^{(p-1)(q-1)} \bar{A}_{q-1}^{\lambda}\right)_{1 \leqslant p, q \leqslant t} . \tag{3.4.13}
\end{equation*}
$$

By applying the block operation $R_{1} \rightarrow R_{1}+R_{2}+\cdots+R_{t}$ and then $R_{i} \rightarrow R_{i}-\frac{1}{t} R_{1}$, $2 \leqslant i \leqslant t$, we see that the numerator of (3.4.12) is
$2 \operatorname{sgn}\left(\sigma_{\lambda}\right) \operatorname{det}\left(\begin{array}{c|c|c|c|c}A_{0}^{\lambda}+\bar{A}_{0}^{\lambda} & 0 & \ldots & 0 & 0 \\ \hline 0 & \omega A_{1}^{\lambda}+\omega^{t-1} \bar{A}_{1}^{\lambda} & \ldots & \omega^{t-2} A_{t-2}^{\lambda}+\omega^{2} \bar{A}_{t-2}^{\lambda} & \omega^{t-1} A_{t-1}^{\lambda}+\omega \bar{A}_{t-1}^{\lambda} \\ \hline 0 & \omega^{2} A_{1}^{\lambda}+\omega^{t-2} \bar{A}_{1}^{\lambda} & \ldots & \omega^{t-4} A_{t-2}^{\lambda}+\omega^{4} \bar{A}_{t-2}^{\lambda} & \omega^{t-2} A_{t-1}^{\lambda}+\omega^{2} \bar{A}_{t-1}^{\lambda} \\ \hline \vdots & \vdots & \ddots & & \vdots \\ \hline 0 & \omega^{t-1} A_{1}^{\lambda}+\omega \bar{A}_{1}^{\lambda} & \ldots & \omega^{2} A_{t-2}^{\lambda}+\omega^{t-2} \bar{A}_{t-2}^{\lambda} & \omega A_{t-1}^{\lambda}+\omega^{t-1} \bar{A}_{t-1}^{\lambda}\end{array}\right)$

This is a $2 \times 2$ block diagonal matrix.

If $\operatorname{core}_{t}(\lambda)$ is not an orthogonal $t$-core, then by Corollary 3.23, either $n_{0}(\lambda) \neq n$ or $n_{i}(\lambda)+n_{t-i}(\lambda) \neq 2 n$ for some $i \in\left[\left\lfloor\frac{t}{2}\right]\right]$. If $n_{0}(\lambda) \neq n$, then the above determinant is 0 by Lemma 3.31. If $n_{0}(\lambda)=n$, then the numerator of (3.4.12) is

$$
\begin{equation*}
2 \operatorname{sgn}\left(\sigma_{\lambda}\right) t^{n} \operatorname{det}\left(A_{0}^{\lambda}+\bar{A}_{0}^{\lambda}\right) \operatorname{det}\left(\left(\omega^{(p-1)(q-1)} A_{q-1}^{\lambda}+\bar{\omega}^{(p-1)(q-1)} \bar{A}_{q-1}^{\lambda}\right)_{2 \leqslant p, q \leqslant t}\right) . \tag{3.4.14}
\end{equation*}
$$

where the last determinant in (3.4.14) is the determinant of $\Pi_{2}$, computed in Corollary 3.38. If $n_{i}(\lambda)+n_{t-i}(\lambda) \neq 2 n$ for some $i \in\left[\left\lfloor\frac{t}{2}\right\rfloor\right]$, then this is 0 by Corollary 3.38(1). In both cases,

$$
\mathrm{o}_{\lambda}^{\text {even }}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=0
$$

If $\operatorname{core}_{t}(\lambda)$ is an orthogonal $t$-core, then by Corollary 3.23, $n_{0}(\lambda)=n$ and $n_{i}(\lambda)+$ $n_{t-i}(\lambda)=2 n, i \in\left[\left\lfloor\frac{t}{2}\right\rfloor\right]$. Using (3.4.14) and Corollary 3.38(2), we see that the numerator of (3.4.12) is

$$
\begin{align*}
2 \operatorname{sgn}\left(\sigma_{\lambda}\right) t^{n} \operatorname{det}\left(A_{0}^{\lambda}+\bar{A}_{0}^{\lambda}\right) & \operatorname{det}\left(\left(\gamma_{i, j}\right)_{1 \leqslant i, j \leqslant t-1}\right)^{n}(-1)^{\Sigma_{2}} \\
& \times \prod_{q=1}^{\left\lfloor\frac{t-1}{2}\right\rfloor} \operatorname{det}\left(\begin{array}{c|c}
A_{q}^{\lambda} & \bar{A}_{t-q}^{\lambda} \\
\hline \bar{A}_{q}^{\lambda} & A_{t-q}^{\lambda}
\end{array}\right) \times \begin{cases}\operatorname{det}\left(A_{\frac{t}{2}}^{\lambda}+\bar{A}_{\frac{t}{2}}^{\lambda}\right) & t \text { even } \\
1 & t \text { odd }\end{cases} \tag{3.4.15}
\end{align*}
$$

The rest of the proof proceeds in almost complete analogy with the proof of Theorem 3.5. Using (3.1.2) and the fact that $\lambda_{t n}=0$ if and only if $\lambda_{n}^{(0)}=0$, we see that the denominator of (3.4.12) is

$$
\begin{align*}
& (-1)^{\frac{t(t-1)}{2} \frac{n(n+1)}{2}} t^{n}\left(1+\delta_{\lambda_{n}^{(0)}, 0}\right) \operatorname{det}\left(A_{0}+\bar{A}_{0}\right) \operatorname{det}\left(\left(\gamma_{i, j}\right)_{1 \leqslant i, j \leqslant t-1}\right)^{n}(-1)^{\Sigma_{2}^{0}} \\
& \times \prod_{q=1}^{\left\lfloor\frac{t-1}{2}\right\rfloor} \operatorname{det}\left(\begin{array}{l|l}
A_{q,-q} & \bar{A}_{t-q, q} \\
\hline \bar{A}_{q,-q} & A_{t-q, q}
\end{array}\right) \times \begin{cases}\operatorname{det}\left(A_{\frac{t}{2}}+\bar{A}_{\frac{t}{2}}\right) & t \text { even } \\
1 & t \text { odd }\end{cases} \tag{3.4.16}
\end{align*}
$$

where

$$
\Sigma_{2}^{0}= \begin{cases}n \sum_{q=1}^{\frac{t-2}{2}} n & t \text { even } . \\ 0 & t \text { odd } .\end{cases}
$$

Taking ratios, we see that one of the factors is exactly the even orthogonal character of
$\lambda^{(0)}$ and the $i$ 'th determinant in the product of (3.4.15) is calculated using

$$
s_{\mu_{i}^{(2)}}\left(X^{t}, \bar{X}^{t}\right)=\frac{(-1)^{\frac{n_{t-i}(\lambda)\left(n_{t-i}(\lambda)-1\right)}{2}}}{(-1)^{\frac{n(n-1)}{2}}} \frac{\operatorname{det}\left(\begin{array}{c|c}
A_{i,-i}^{\lambda} & \bar{A}_{t-i, i}^{\lambda}  \tag{3.4.17}\\
\hline \bar{A}_{i,-i}^{\lambda} & A_{t-i, i}^{\lambda}
\end{array}\right)}{\operatorname{det}\left(\begin{array}{c|c}
A_{i,-i} & \bar{A}_{t-i, i} \\
\hline \bar{A}_{i,-i} & A_{t-i, i}
\end{array}\right),}
$$

where $\mu_{i}^{(2)}=\lambda_{1}^{(t-i)}+\left(\lambda^{(i)}, 0, \ldots, 0,-\operatorname{rev}\left(\lambda^{(t-i)}\right)\right)$. The only new part is the final determinant, which is calculated using (3.2.14) and (2.4.8), and we get

Finally, the even orthogonal character is given by

$$
\begin{aligned}
& \mathrm{o}_{\lambda}^{\text {even }}\left(X, \omega X, \ldots, \omega^{t-1} X\right) \\
& \quad=(-1)^{\epsilon} \operatorname{sgn}\left(\sigma_{\lambda}\right) \mathrm{o}_{\lambda(0)}^{\text {even }}\left(X^{t}\right) \prod_{i=1}^{\left\lfloor\frac{t-1}{2}\right\rfloor} s_{\mu_{i}^{(2)}}\left(X^{t}, \bar{X}^{t}\right) \times \begin{cases}(-1)^{\sum_{i=1}^{n} \lambda_{i}^{(t / 2)}} \mathrm{So}_{\lambda}(t / 2)\left(-X^{t}\right) & t \text { even }, \\
1 & t \text { odd },\end{cases}
\end{aligned}
$$

where

$$
\begin{aligned}
\epsilon=\frac{t(t-1)}{2} \frac{n(n+1)}{2}+\sum_{q=1}^{\left\lfloor\frac{t-1}{2}\right\rfloor}\left(n_{q}(\lambda)-n\right) & \times \begin{cases}n & t \text { even } \\
0 & t \text { odd }\end{cases} \\
& +\sum_{i=1}^{\left\lfloor\frac{t-1}{2}\right\rfloor}\left(\frac{n(n-1)}{2}-\frac{n_{t-i}(\lambda)\left(n_{t-i}(\lambda)-1\right)}{2}\right) .
\end{aligned}
$$

After similar simplifications, the parity of $\epsilon$ shown to be the same as

$$
-\sum_{i=\left\lfloor\frac{t+2}{2}\right\rfloor}^{t-1}\binom{n_{i}(\lambda)}{2}+ \begin{cases}\frac{n(n+t-1)}{2}+n r & t \text { even } \\ \frac{(t-1) n}{2} & t \text { odd }\end{cases}
$$

where $r$ is the rank by Lemma 3.28(2), completing the proof.

### 3.4.3 Odd orthogonal characters

Recall the matrices $A_{p, q}^{\lambda}$ and $\bar{A}_{p, q}^{\lambda}$ from (3.2.9). Consider the $t n \times t n$ block matrix

$$
\Pi_{3}=\left(\omega^{(p-1) q} A_{q-1,1}^{\lambda}-\bar{\omega}^{(p-1)(q-1)} \bar{A}_{q-1,0}^{\lambda}\right)_{1 \leqslant p, q \leqslant t} .
$$

Substitution of $U_{j}=A_{j-1,1}^{\lambda}, V_{j}=\bar{A}_{j-1,0}^{\lambda}$ for $1 \leqslant j \leqslant t$ and

$$
\gamma_{i, j}= \begin{cases}\omega^{\frac{(i-1)(j+1)}{2}} & j \text { odd } \\ \omega^{-\frac{(i-1)(j-2)}{2}} & j \text { even }\end{cases}
$$

in Lemma 3.32 proves the following corollary.
Corollary 3.39. 1. If $n_{i}(\lambda)+n_{t-1-i}(\lambda) \neq 2 n$ for some $i \in\left[0,\left\lfloor\frac{t-1}{2}\right\rfloor\right]$, then $\operatorname{det} \Pi_{3}=0$.
2. If $n_{i}(\lambda)+n_{t-1-i}(\lambda)=2 n$ for all $i \in\left\{0,1, \ldots,\left\lfloor\frac{t-1}{2}\right\rfloor\right\}$, then

$$
\begin{align*}
& \operatorname{det} \Pi_{3}=\left(\operatorname{det}\left(\gamma_{i, j}\right)_{1 \leqslant i, j \leqslant t}\right)^{n}(-1)^{\Sigma_{3}} \prod_{q=1}^{\left\lfloor\frac{t}{2}\right\rfloor} \operatorname{det}\left(\begin{array}{l|l}
A_{q-1,1}^{\lambda} & \bar{A}_{t-q, 0}^{\lambda} \\
\hline \bar{A}_{q-1,0}^{\lambda} & A_{t-q, 1}^{\lambda}
\end{array}\right)  \tag{3.4.19}\\
& \times \begin{cases}\operatorname{det}\left(A_{\frac{t-1}{2}, 1}^{\lambda}-\bar{A}_{\frac{t-1}{2}, 0}^{\lambda}\right) & t \text { odd }, \\
1 & t \text { even },\end{cases}
\end{align*}
$$

where

$$
\Sigma_{3}=\sum_{q=1}^{\left\lfloor\frac{t}{2}\right\rfloor}\left(n+n_{q-1}(\lambda)\right)+ \begin{cases}n \sum_{q=1}^{\frac{t-1}{2}} n_{q-1}(\lambda) & t \text { odd } \\ 0 & t \text { even } .\end{cases}
$$

Proof of Theorem 3.11. Starting from the formula for the odd orthogonal character in (2.4.1), we see that the desired polynomial is

$$
\begin{equation*}
\operatorname{so}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=\frac{\operatorname{det}\left(\left(\left(\omega^{p-1} x_{i}\right)^{\beta_{j}(\lambda)+1}-\left(\bar{\omega}^{p-1} \bar{x}_{i}\right)^{\beta_{j}(\lambda)}\right)_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant t n}}\right)_{1 \leqslant p \leqslant t}}{\operatorname{det}\left(\left(\left(\omega^{p-1} x_{i}\right)^{t n-j+1}-\left(\bar{\omega}^{p-1} \bar{x}_{i}\right)^{t n-j}\right)_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant t n}}\right)_{1 \leqslant p \leqslant t}} . \tag{3.4.20}
\end{equation*}
$$

We again proceed as in the proof of Theorem 3.5. Permuting the columns of the determinant in the numerator in (3.4.20) by the permutation $\sigma_{\lambda}$ from (3.1.1), we see that the numerator in 3.4 .20 is

$$
\begin{equation*}
\operatorname{sgn}\left(\sigma_{\lambda}\right) \operatorname{det}\left(\omega^{(p-1) q} A_{q-1,1}^{\lambda}-\bar{\omega}^{(p-1)(q-1)} \bar{A}_{q-1}^{\lambda}\right)_{1 \leqslant p, q \leqslant t}, \tag{3.4.21}
\end{equation*}
$$

where the last determinant in (3.4.14) is the determinant of $\Pi_{2}$, computed in Corollary 3.39. If core ${ }_{t}(\lambda)$ is not self-conjugate, then by Corollary 3.18, $n_{i}(\lambda)+n_{t-1-i}(\lambda) \neq 2 n$ for some $i \in\left\{0,1, \ldots,\left\lfloor\frac{t-1}{2}\right\rfloor\right\}$. In that case, the determinant is 0 by Corollary 3.39(1). Hence

$$
\operatorname{so}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=0
$$

If $\operatorname{core}_{t}(\lambda)$ is self-conjugate, then by Corollary 3.18, $n_{i}(\lambda)+n_{t-1-i}(\lambda)=2 n$ for all $i \in\left\{0,1, \ldots,\left\lfloor\frac{t-1}{2}\right\rfloor\right\}$. By Corollary 3.39 (2), the numerator in (3.4.20) is
$\operatorname{sgn}\left(\sigma_{\lambda}\right)\left(\operatorname{det}\left(\gamma_{i, j}\right)_{1 \leqslant i, j \leqslant t}\right)^{n}(-1)^{\Sigma_{3}} \prod_{q=1}^{\left\lfloor\frac{t}{2}\right\rfloor} \operatorname{det}\left(\begin{array}{ll|}A_{q-1,1}^{\lambda} & \bar{A}_{t-q}^{\lambda} \\ \hline \bar{A}_{q-1}^{\lambda} & A_{t-q, 1}^{\lambda}\end{array}\right) \times \begin{cases}1 & t \text { even, } \\ \operatorname{det}\left(A_{\frac{t-1}{2}, 1}^{\lambda}-\bar{A}_{\frac{t-1}{2}}^{\lambda}\right) & t \text { odd. }\end{cases}$
We now evaluate the $2 \times 2$ block determinant as follows: for $1 \leqslant q \leqslant\left\lfloor\frac{t}{2}\right\rfloor$, we multiply row $i$ in the top blocks by $\bar{x}_{i}^{q}$ and row $i$ in the bottom blocks by $x_{i}^{q-1}$, for each $i$. We then end up with

$$
\begin{align*}
\operatorname{sgn}\left(\sigma_{\lambda}\right)\left(\operatorname{det}\left(\gamma_{i, j}\right)_{1 \leqslant i, j \leqslant t}\right)^{n}(-1)^{\Sigma_{3}} \prod_{q=1}^{\left\lfloor\frac{t}{2}\right\rfloor}\left(\begin{array}{l|l}
\bar{x}_{1} \bar{x}_{2} \ldots & \left.\bar{x}_{n} \operatorname{det}\left(\begin{array}{ll}
A_{q-1,1-q}^{\lambda} & \bar{A}_{t-q, q}^{\lambda} \\
\hline \bar{A}_{q-1,1-q}^{\lambda} & A_{t-q, q}^{\lambda}
\end{array}\right)\right) \\
\times & \begin{cases}1 & t \text { even } \\
\operatorname{det}\left(A_{\frac{t-1}{2}, 1}^{\lambda}-\bar{A}_{\frac{t-1}{2}}^{\lambda}\right) & t \text { odd }\end{cases}
\end{array} .\right. \tag{3.4.23}
\end{align*}
$$

The denominator in (3.4.20) is therefore

$$
\begin{align*}
& (-1)^{\frac{t(t-1)}{2} \frac{n(n+1)}{2}}\left(\operatorname{det}\left(\gamma_{i, j}\right)_{1 \leqslant i, j \leqslant t}\right)^{n}(-1)^{\Sigma_{3}^{0}} \prod_{q=1}^{\left\lfloor\frac{t}{2}\right\rfloor}\left(\bar{x}_{1} \bar{x}_{2} \ldots \bar{x}_{n} \operatorname{det}\left(\begin{array}{c|c}
A_{q-1,1-q} & \bar{A}_{t-q, q} \\
\hline \bar{A}_{q-1,1-q} & A_{t-q, q}
\end{array}\right)\right) \\
& \times \begin{cases}1 & t \text { even, } \\
\operatorname{det}\left(A_{\frac{t-1}{2}, 1}-\bar{A}_{\frac{t-1}{2}}\right) & t \text { odd, }\end{cases} \tag{3.4.24}
\end{align*}
$$

where

$$
\Sigma_{3}^{0}=\sum_{q=1}^{\left\lfloor\frac{t}{2}\right\rfloor} 2 n+ \begin{cases}n \sum_{q=1}^{\frac{t-1}{2}} n & t \text { odd } \\ 0 & t \text { even }\end{cases}
$$

Taking ratios, we see that the block determinants are proportional to Schur functions
using Lemma 3.29,

$$
s_{\mu_{i}^{(3)}}\left(X^{t}, \bar{X}^{t}\right)=\frac{(-1)^{\frac{n_{t-1-i}(\lambda)\left(n_{t-1-i}(\lambda)-1\right)}{2}}}{(-1)^{\frac{n(n-1)}{2}}} \frac{\operatorname{det}\left(\begin{array}{c|c}
A_{i,-i}^{\lambda} & \bar{A}_{t-1-i, i+1}^{\lambda}  \tag{3.4.25}\\
\hline \bar{A}_{i,-i}^{\lambda} & A_{t-1-i, i+1}^{\lambda}
\end{array}\right)}{\operatorname{det}\left(\begin{array}{c|c}
A_{i,-i} & \bar{A}_{t-i, i} \\
\hline \bar{A}_{i,-i} & A_{t-i, i}
\end{array}\right)},
$$

where $\mu_{i}^{(3)}=\lambda_{1}^{(t-1-i)}+\left(\lambda^{(i)}, 0, \ldots, 0,-\operatorname{rev}\left(\lambda^{(t-1-i)}\right)\right)$. The last ratio of determinants gives an odd orthogonal character. Finally, the odd orthogonal character is given by

$$
\operatorname{so}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=(-1)^{\epsilon} \operatorname{sgn}\left(\sigma_{\lambda}\right) \prod_{i=0}^{\left\lfloor\frac{t-2}{2}\right\rfloor} s_{\mu_{i}^{(3)}}\left(X^{t}, \bar{X}^{t}\right) \times \begin{cases}\text { so }_{\lambda}\left(\frac{t-1}{2}\right) \\ 1 & t \text { odd } \\ 1 & t \text { even }\end{cases}
$$

where

$$
\begin{aligned}
\epsilon=\frac{t(t-1)}{2} \frac{n(n+1)}{2}+\sum_{q=0}^{\left\lfloor\frac{t-2}{2}\right\rfloor}\left(n_{q}(\lambda)\right. & -n) \times \begin{cases}n+1 & t \text { odd, } \\
1 & t \text { even, }\end{cases} \\
& +\sum_{i=0}^{\left\lfloor\frac{t-2}{2}\right\rfloor}\left(\frac{n(n-1)}{2}-\frac{n_{t-1-i}(\lambda)\left(n_{t-1-i}(\lambda)-1\right)}{2}\right) .
\end{aligned}
$$

After similar simplifications, $\epsilon$ turns out to have the same parity as

$$
-\sum_{i=\left\lfloor\frac{t+1}{2}\right\rfloor}^{t-1}\binom{n_{i}(\lambda)+1}{2}+ \begin{cases}n r & t \text { odd } \\ 0 & t \text { even }\end{cases}
$$

where $r$ is the rank by Lemma 3.28(3), completing the proof.

### 3.5 Generating functions

We now give enumerative results for $z$-asymmetric partitions defined in Definition 3.3. We first recall that the $q$-Pochhammer symbol is given by

$$
\begin{equation*}
(a ; q)_{m}=\prod_{j=0}^{m-1}\left(1-a q^{j}\right), \tag{3.5.1}
\end{equation*}
$$

so that $(a ; q)_{0}=1$. We also define the limiting infinite product

$$
\begin{equation*}
(a ; q)_{\infty}=\prod_{j=0}^{\infty}\left(1-a q^{j}\right) . \tag{3.5.2}
\end{equation*}
$$

Many generating functions in the theory of partitions can be naturally expressed in terms of $q$-Pochhammer symbols. For example, the generating function for all partitions is $1 /(q ; q)_{\infty}$ and that of strict partitions is $(-q ; q)_{\infty}$.

Proposition 3.40. The number of $z$-asymmetric partitions of $m$ is equal to the number of partitions of $m$ with distinct parts of the form $2 k+1+z, k \geqslant 0$.

Proof. To prove the proposition, we construct a bijection from the set $\mathcal{P}_{z}$ to the set of partitions of $n$ with distinct parts of the form $2 k+1+z, k \geqslant 0$. If $\lambda=(\alpha \mid \alpha+z)$ is a $z$-symmetric partition of rank $r$, then define $\mu$ of length $r$ by $\mu_{i}=2 \alpha_{i}+z+1$. Then all the parts of $\mu$ are distinct and of the desired form. This map is clearly invertible.

Proposition 3.40 immediately gives an expression of the generating function for $z$-asymmetric partitions.

Corollary 3.41. For $z \in \mathbb{Z}$,

$$
\sum_{\lambda \in \mathcal{P}_{z}} q^{|\lambda|}=\prod_{k \geqslant 0}\left(1+q^{z+1+2 k}\right)=\left(-q^{z+1} ; q^{2}\right)_{\infty} .
$$

We now move on to enumerating $z$-asymmetric partitions which are also $t$-cores. Recall from Lemma 3.20 that there are no nontrivial partitions if $z>t-2$.

Theorem 3.42. Let $z \leqslant t-2$. Represent elements of $\mathbb{Z}^{\left\lfloor\frac{t-z}{2}\right\rfloor}$ by $\left(z_{0}, \ldots, z_{\left\lfloor\frac{t-z-2}{2}\right\rfloor}\right)$ and define $b \in \mathbb{Z}^{\left\lfloor\frac{t-z}{2}\right\rfloor}$ by $\vec{b}_{i}=t-z-1-2 i$. Then there exists a bijection $\phi: \mathcal{P}_{z, t} \rightarrow \mathbb{Z}^{\left\lfloor\frac{t-z}{2}\right\rfloor}$ satisfying $|\lambda|=t\|\phi(\vec{\lambda})\|^{2}-\vec{b} \cdot \phi(\vec{\lambda})$, where . represents the standard inner product.

Proof. Suppose $\lambda \in \mathcal{P}_{z, t}$, of length at most $t n$ for some $n \geqslant 1$. Define the map $\phi$ by

$$
(\phi(\lambda))_{i}:=n_{i}(\lambda)-n, \quad 0 \leqslant i \leqslant\left\lfloor\frac{t-z-2}{2}\right\rfloor .
$$

Since $n$ is not unique, it is not a priori clear that $\phi$ is well-defined. But from the definition of $n_{i}(\lambda)$, it is easy to see that $n_{i}(\lambda)-n=n_{i}(\lambda, t n+t)-n-1$. Hence, $\phi(\lambda)$ is indeed well-defined.

To show that $\phi$ is a bijection, we define the inverse of $\phi$ as follows. For a vector $\vec{v}=\left(v_{0}, v_{1}, \ldots, v_{\left\lfloor\frac{t-z-2}{2}\right\rfloor}\right)$, let $n=\max \left\{\left|v_{0}\right|,\left|v_{1}\right|, \ldots,\left|v_{\left\lfloor\frac{t-z-2}{2}\right\rfloor}\right|\right\}$ and for $0 \leqslant i \leqslant t-1$,

$$
m_{i}= \begin{cases}n+v_{i} & 0 \leqslant i \leqslant\left\lfloor\frac{t-z-2}{2}\right\rfloor \\ n-v_{t-z-1-i} & \left\lfloor\frac{t-z+1}{2}\right\rfloor \leqslant i \leqslant t-z-1, \\ n & \text { otherwise }\end{cases}
$$

By construction, $\sum_{i=0}^{t-1} m_{i}=t n, m_{i}+m_{t-z-1-i}=2 n$ for $0 \leqslant i \leqslant\left\lfloor\frac{t-z-1}{2}\right\rfloor, m_{i}=n$ for $t-z \leqslant i \leqslant t-1$. By Lemma 3.21, there is a unique $t$-core $\lambda \in \mathcal{P}_{z, t}$ satisfying $n_{i}(\lambda)=m_{i}$. and we set $\phi^{-1}(\vec{v})=\lambda$. Moreover the size of $\lambda$ is computed as

$$
\begin{equation*}
|\lambda|=\sum_{i=1}^{t n} \beta_{i}(\lambda)-\frac{t n(t n-1)}{2} . \tag{3.5.3}
\end{equation*}
$$

Since $\lambda$ is a $t$-core, $t j+i, 0 \leqslant j \leqslant n_{i}(\lambda)-1,0 \leqslant i \leqslant t-1$ are the parts of $\beta(\lambda)$ (see Proposition 2.3). So,

$$
\begin{aligned}
& \sum_{i=1}^{t n} \beta_{i}(\lambda)=\sum_{i=0}^{t-1}\left(i\left(n_{i}(\lambda)\right)+\frac{n_{i}(\lambda)\left(n_{i}(\lambda)-1\right) t}{2}\right) \\
& \quad=\sum_{i=0}^{t-1}\left(i\left(n_{i}(\lambda)-n\right)\right)+\frac{t n(t-1)}{2}+\frac{t}{2} \sum_{i=0}^{t-1} n_{i}(\lambda)^{2}-\frac{t^{2} n}{2} .
\end{aligned}
$$

Substituting this in (3.5.3), we get

$$
\begin{aligned}
|\lambda|=\sum_{i=0}^{t-1}\left(i\left(n_{i}(\lambda)-n\right)\right)+\frac{t}{2}\left(\sum_{i=0}^{t-1} n_{i}(\lambda)^{2}-t n^{2}\right) & \\
& =\sum_{i=0}^{t-1}\left(i\left(n_{i}(\lambda)-n\right)\right)+\frac{t}{2} \sum_{i=0}^{t-1}\left(n_{i}(\lambda)-n\right)^{2} .
\end{aligned}
$$

Now observe that

$$
-\vec{b} \cdot \vec{v}=\sum_{i=0}^{\left\lfloor\frac{t-z-2}{2}\right\rfloor}(z+1-t+2 i) v_{i}=\sum_{i=0}^{\left\lfloor\frac{t-z-2}{2}\right\rfloor}(z+1-t+2 i)\left(n_{i}(\lambda)-n\right) .
$$

Since $\lambda \in \mathcal{P}_{z, t}$, using Lemma 3.21, we have

$$
\begin{aligned}
-\vec{b} \cdot \vec{v} & =\sum_{i=0}^{t-1} i\left(n_{i}(\lambda)-n\right), \\
\sum_{i=0}^{t-1}\left(n_{i}(\lambda)-n\right)^{2} & =2 \sum_{i=0}^{\left\lfloor\frac{t-z-2}{2}\right\rfloor}\left(n_{i}(\lambda)-n\right)^{2}=2\|\vec{v}\|^{2} .
\end{aligned}
$$

Hence $|\lambda|=t\|\vec{v}\|^{2}-\vec{b} \cdot \vec{v}$, completing the proof.
Define the Ramanujan theta function [23, Equation (18.1)],

$$
\begin{equation*}
f(a, b)=\sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \tag{3.5.4}
\end{equation*}
$$

which is related to the Jacobi theta function. We consider $f(a, b)$ to be an element of the ring of formal power series $\mathbb{Z}[[a, b]]$. There are several nice identities satisfied by $f$. For example, $f(a, b)=f(b, a), f(1, a)=2 f\left(a, a^{3}\right)$ and $f(-1, a)=0$ [23, Chapter 16, Entry 18]. In addition, because of the Jacobi triple product identity, we have [23, Chapter 16, Entry 19],

$$
f(a, b)=(-a ; a b)_{\infty}(-b ; a b)_{\infty}(a b ; a b)_{\infty} .
$$

Let $p_{z, t}(m)$ be the cardinality of partitions in $\mathcal{P}_{z, t}$ of size $m$.
Corollary 3.43. For $z \leqslant t-2$, we have

$$
\sum_{m \geqslant 0} p_{z, t}(m) q^{m}=\prod_{i=0}^{\lfloor(t-z-2) / 2\rfloor} f\left(q^{2 i+z+1}, q^{2 t-2 i-z-1}\right)
$$

Proof. As a consequence of Theorem 3.42,

$$
\sum_{m \geqslant 0} p_{z, t}(m) q^{m}=\sum_{\vec{v} \in \mathbb{Z}\left\lfloor\frac{t-z}{2}\right\rfloor} \prod_{i=0}^{\left\lfloor\frac{t-z-2}{2}\right\rfloor} q^{t v_{i}^{2}-(t-z-1-2 i) v_{i}}
$$

Rewriting the exponent and interchanging the order of summation, we see that the generating function becomes

$$
\prod_{i=0}^{\left\lfloor\frac{t-z-2}{2}\right\rfloor} \sum_{v_{i} \in \mathbb{Z}} q^{(2 i+z+1) \frac{v_{i}\left(v_{i}+1\right)}{2}+(2 t-2 i-z-1) \frac{v_{i}\left(v_{i}-1\right)}{2}}=\prod_{i=0}^{\lfloor(t-z-2) / 2\rfloor} f\left(q^{2 i+z+1}, q^{2 t-2 i-z-1}\right)
$$

completing the proof.

We remark that the special case of $z=0$ (i.e. self-conjugate $t$-cores) in Corollary 3.43 was obtained by Garvan-Kim-Stanton [41, Equations (7.1a) and (7.1b)]. Thus, our result can be viewed as a generalization of theirs for symplectic and orthogonal partitions, leading to an immediate proof of Theorem 3.15.

## Chapter 4

## Factorization of universal characters twisted by roots of unity

In this chapter, we give different proofs of the factorization results of other classical characters Theorem 3.5, Theorem 3.9 and Theorem 3.11) using the Jacobi-Trudi identities. Chen, Garsia and Remmel [28] gave an alternate proof of the Schur factorization result Theorem 3.1) based on the Jacobi-Trudi identity. Recently using a similar proof strategy, Albion [2] lifted all the factorization results to the level of universal characters. In Section 4.1, we prove some determinantal identities. We give alternate proofs of the factorization results of other classical characters in Section 4.2,

### 4.1 Background results

For $r \in \mathbb{Z}$, define

$$
u_{r}\left(x_{1}, \ldots, x_{n}\right)=\sum_{i=1}^{n} \frac{x_{i}^{n+r-1}}{\prod_{j=1, j \neq i}^{n}\left(x_{i}-x_{j}\right)}
$$

The following Lemma expresses $u_{r}\left(x_{1}, \ldots, x_{n}\right)$ in terms of the complete symmetric functions. It is the $q=1$ case of classical Bernstein operator,

$$
B_{r}(x ; q)=\sum_{i=1}^{n} \frac{x_{i}^{n+r-1}}{\prod_{j=1, j \neq i}^{n}\left(x_{i}-x_{j}\right)} T_{q, x_{i}}
$$

where $\left(T_{q, x_{i}} f\right)\left(x_{1}, \ldots, x_{n}\right)=f\left(x_{1}, \ldots, x_{i-1}, q x_{i}, x_{i+1}, \ldots, x_{n}\right)$, which can be used to construct the modified Hall-Littlewood polynomials [55].

Lemma 4.1. For $r \in \mathbb{Z}$,

$$
u_{r}(X)= \begin{cases}h_{r}(X) & r>-n \\ \frac{(-1)^{n-1}}{x_{1} \ldots x_{n}} h_{-r-n}(\bar{X}) & r \leqslant-n\end{cases}
$$

Given an $n$-tuple $\underline{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{n}\right)_{>} \in \mathbb{Z}^{n}$, let

$$
\begin{align*}
& \quad V_{\underline{\alpha}}\left(x_{1}, \ldots, x_{n}\right)=\operatorname{det}_{1 \leqslant i, j \leqslant n}\left(x_{i}^{\alpha_{j}+n-j}\right) \\
& \text { and } \quad \mathcal{U}_{\underline{\alpha}}\left(x_{1}, \ldots, x_{n}\right)=\operatorname{det}_{1 \leqslant i, j \leqslant n}\left(u_{\alpha_{i}-i+j}(X)\right) . \tag{4.1.1}
\end{align*}
$$

Remark 4.2. If $\underline{\alpha}$ is a partition, then $\mathcal{U}_{\underline{\alpha}}\left(x_{1}, \ldots, x_{n}\right)=s_{\underline{\alpha}}\left(x_{1}, \ldots, x_{n}\right)$.
Theorem 4.3. For an $n$-tuple $\underline{\alpha}=\left(\alpha_{1}, \ldots, \alpha_{n}\right)_{>} \in \mathbb{Z}^{n}$,

$$
\mathcal{U}_{\underline{\alpha}}\left(x_{1}, \ldots, x_{n}\right)=\frac{V_{\underline{\alpha}}\left(x_{1}, \ldots, x_{n}\right)}{V_{\underline{0}}\left(x_{1}, \ldots, x_{n}\right)} .
$$

Proof. We note that $u_{r}\left(x_{1}, \ldots, x_{n-1}\right)=u_{r}\left(x_{1}, \ldots, x_{n}\right)-x_{n} u_{r-1}\left(x_{1}, \ldots, x_{n}\right)$. The proof of the theorem follows from the similar ideas as in [16, Theorem 2] with $H_{\mathbf{k}}\left(x_{1}, \ldots, x_{n}\right)$ replaced by $\mathcal{U}_{\underline{\alpha}}$.

Corollary 4.4. For $k \in \mathbb{Z}$, we have

$$
\mathcal{U}_{\underline{\alpha+k}}\left(x_{1}, \ldots, x_{n}, \bar{x}_{1}, \ldots, \bar{x}_{n}\right)=\mathcal{U}_{\underline{\alpha}}\left(x_{1}, \ldots, x_{n}, \bar{x}_{1}, \ldots, \bar{x}_{n}\right) .
$$

Lemma 4.5. Let $\lambda$ be a partition of length at most tn. If $p, q \in\{0,1, \ldots, t-1\}$ such that $n_{p}(\lambda)+n_{q}(\lambda)=2 n$, then we define $\rho_{p, q}=\lambda_{1}^{(p)}+\left(\lambda^{(q)}, 0, \ldots, 0,-\operatorname{rev}\left(\lambda^{(p)}\right)\right)$, where we pad $0^{\prime} s$ in the middle so that $\rho_{p, q}$ is of length $2 n$. Then the Schur function $s_{\rho_{p, q}}\left(X^{t}, \bar{X}^{t}\right)$ can be written as

$$
=\frac{(-1)^{\frac{n_{p}(\lambda)\left(n_{p}(\lambda)+1\right)}{2}}}{(-1)^{\frac{n(n+1)}{2}}} \operatorname{det}\left(\begin{array}{c|c}
\left(h_{\beta_{p}(\lambda(q))-n+j}(X, \bar{X})\right)_{\substack{t \leqslant i \leqslant n_{q} \\
1 \leqslant j \leqslant n}} & \left(h_{\beta_{i}(\lambda(q))-n-j+1}(X, \bar{X})\right)_{\substack{1 \leqslant i \leqslant n_{q} \\
1 \leqslant j \leqslant n}} \\
\hline\left(h_{\beta_{i}(\lambda(p))-n-j+1}(X, \bar{X})\right)_{\substack{1 \leqslant i \leqslant n_{p} \\
1 \leqslant j \leqslant n}} & \left(h_{\beta_{i}(\lambda(p))-n+j}(X, \bar{X})\right)_{\substack{1 \leqslant i \leqslant n_{p} \\
1 \leqslant j \leqslant n}} \tag{4.1.2}
\end{array}\right) .
$$

## Corollary 4.6.

$$
\lambda_{1}^{(p)}+\beta_{i}\left(\lambda^{(q)}\right)-n_{q}+j-\lambda_{1}^{(p)}-n_{p}
$$

$$
-\left(\lambda_{1}^{(p)}-\beta_{n_{p}+1-i}^{(p)}(\lambda)-n_{p}+j-1\right)+\left(\lambda_{1}^{(p)}-n_{p}\right)
$$

$\operatorname{det}\left(\begin{array}{c|c}\begin{array}{c}\left(h_{\lambda_{1}^{(p)}+\beta_{i}(\lambda(q))-n_{q}+j}(X, \bar{X})\right)_{\substack{1 \leqslant i \leqslant n_{q} \\ 1 \leqslant j \leqslant n}}\end{array} \overbrace{\lambda_{1}^{(p)}+\beta_{i}\left(\lambda^{(q)}\right)-n_{q}+n+j+1}(X, \bar{X}))_{\substack{1 \leqslant i \leqslant n_{q} \\ 1 \leqslant j \leqslant n}} \\ \hline\left(h_{\lambda_{1}^{(p)}-\beta_{n_{p}+1-i}^{(p)}(\lambda)-n_{p}+j-1}(X, \bar{X})\right)_{\substack{1 \leqslant i \leqslant n_{p} \\ 1 \leqslant j \leqslant n}} & \left(h_{\lambda_{1}^{(p)}-\beta_{n_{p}+1-i}^{(p)}(\lambda)-n_{p}+j+n-1}(X, \bar{X})\right)_{\substack{1 \leqslant i \leqslant n_{p} \\ 1 \leqslant j \leqslant n}}\end{array}\right)$.

Proof. Using Lemma 4.1, we note that

$$
U_{r}:=u_{r}(X, \bar{X})= \begin{cases}h_{r}(X, \bar{X}) & r>-2 n  \tag{4.1.3}\\ -h_{-r-2 n}(X, \bar{X}) & r \leqslant-2 n\end{cases}
$$

Since the complete symmetric functions in the determinant in the right hand side of (4.1.2) are indexed by integers greater than $-2 n$, using 4.1.3), we see that the determinant in right hand side of (4.1.2) is

$$
\operatorname{det}\left(\begin{array}{c|c}
\left(U_{\beta_{i}(\lambda(q))-n+j}\right)_{\substack{1 \leqslant i \leqslant n_{q} \\
1 \leqslant j \leqslant n}} & \left(U_{\beta_{i}(\lambda(q))-n-j+1}\right)_{\substack{1 \leqslant i \leqslant n_{q} \\
1 \leqslant j \leqslant n}} \\
\hline\left(U_{\beta_{i}(\lambda(p))-n-j+1}\right)_{\substack{1 \leqslant i \leqslant n_{p} \\
1 \leqslant j \leqslant n}} & \left(U_{\beta_{i}\left(\lambda^{(p)}\right)-n+j}\right)_{\substack{1 \leqslant i \leqslant n_{p} \\
1 \leqslant j \leqslant n}}
\end{array}\right) .
$$

Substitution of $U_{r}=-U_{-r-2 n}, r \geqslant-2 n$ in bottom blocks gives

$$
\operatorname{det}\left(\begin{array}{c|c}
{\left(U _ { \beta _ { i } ( \lambda ( q ) } \left(\lambda^{\prime}-n+j\right.\right.}^{)_{\substack{1 \leqslant i \leqslant n_{q} \\
1 \leqslant j \leqslant n}}} & \left(U_{\beta_{i}(\lambda(q))-n-j+1}\right)_{\substack{1 \leqslant i \leqslant n_{q} \\
1 \leqslant j \leqslant n}} \\
\hline\left(-U_{-\beta_{i}(\lambda(p))-n+j-1}\right)_{\substack{1 \leqslant i \leqslant n_{p} \\
1 \leqslant j \leqslant n}} & \left(-U_{-\beta_{i}(\lambda(p))-n-j}\right)_{\substack{1 \leqslant i \leqslant n_{p} \\
1 \leqslant j \leqslant n}}
\end{array}\right) .
$$

Applying $C_{j} \leftrightarrow C_{n+j}$, for all $j \in[n]$, we get

$$
(-1)^{n} \operatorname{det}\left(\begin{array}{c|c}
\left(U_{\beta_{i}(\lambda(q))-n-j+1}\right)_{\substack{1 \leqslant i \leqslant n_{q} \\
1 \leqslant j \leqslant n}} & \left(U_{\beta_{i}(\lambda(q))-n+j}\right)_{\substack{1 \leqslant i \leqslant n_{q} \\
1 \leqslant j \leqslant n}} \\
\hline\left(-U_{\left.-\beta_{i}(\lambda(p))-n-j\right)}\right)_{\substack{1 \leqslant i \leqslant n_{p} \\
1 \leqslant j \leqslant n}} & \left(-U_{-\beta_{i}(\lambda(p))-n+j-1}\right)_{\substack{1 \leqslant i \leqslant n_{p} \\
1 \leqslant j \leqslant n}}
\end{array}\right) .
$$

Reversing the last $n_{p}$ rows and then the first $n$ columns, we have

$$
\frac{(-1)^{\frac{n_{p}\left(n_{p}+1\right)}{2}}}{(-1)^{\frac{n(n+1)}{2}}} \operatorname{det}\left(\begin{array}{c|c}
\left(U_{\beta_{i}(\lambda(q))-2 n+j}\right)_{\substack{1 \leqslant i \leqslant n_{q} \\
1 \leqslant j \leqslant n}} & \left(U_{\beta_{i}(\lambda(q))-n+j}\right)_{\substack{1 \leqslant i \leqslant n_{q} \\
1 \leqslant j \leqslant n}} \\
\hline\left(U_{-\beta_{n_{p}+1-i}(\lambda(p))-2 n+j-1}\right)_{\substack{1 \leqslant i \leqslant n_{p} \\
1 \leqslant j \leqslant n}} & \left(U_{-\beta_{n_{p}+1-i}\left(\lambda^{(p)}\right)-n+j-1}\right)_{\substack{1 \leqslant i \leqslant n_{p} \\
1 \leqslant j \leqslant n}}
\end{array}\right) .
$$

Using (4.1.1) and then by Corollary 4.4, we see that the determinant is

$$
\begin{aligned}
\frac{(-1)^{\frac{n_{p}\left(n_{p}+1\right)}{2}}}{(-1)^{\frac{n(n+1)}{2}}} \mathcal{U}_{\left(\lambda^{(q)}-n_{p}, 0, \ldots, 0,-\operatorname{rev}\left(\lambda^{(p)}\right)-n_{p}\right)} & (X, \bar{X}) \\
& =\frac{(-1)^{\frac{n_{p}\left(n_{p}+1\right)}{2}}}{(-1)^{\frac{n(n+1)}{2}}} \mathcal{U}_{\lambda_{1}^{(p)}+\left(\lambda^{(q)}, 0, \ldots, 0,-\operatorname{rev}\left(\lambda^{(p)}\right)\right)}(X, \bar{X}) .
\end{aligned}
$$

Using Remark 4.2 completes the proof.
Lemma 4.7. For a partition of length at most $n$, the odd orthogonal character $\mathrm{so}_{\lambda}(X)$ is given by

$$
\operatorname{so}_{\lambda}(X)=\operatorname{det}\left(h_{\beta_{i}(\lambda)-n+j}(X, \bar{X})+h_{\beta_{i}(\lambda)-n-j+1}(X, \bar{X})\right)_{1 \leqslant i, j \leqslant n} .
$$

Proof. Using (2.4.9), we see that the odd orthogonal character is

$$
\begin{align*}
\operatorname{so}_{\lambda}(X)= & \operatorname{det}\left(h_{\lambda_{i}-i+j}(X, \bar{X}, 1)-h_{\lambda_{i}-i-j}(X, \bar{X}, 1)\right)_{1 \leqslant i, j \leqslant n} \\
= & \operatorname{det}\left(\sum_{p=0}^{\lambda_{i}-i+j} h_{p}(X, \bar{X})-\sum_{p=0}^{\lambda_{i}-i-j} h_{p}(X, \bar{X})\right)_{1 \leqslant i, j \leqslant n}  \tag{4.1.4}\\
& =\operatorname{det}\left(\sum_{p=\lambda_{i}-i-j+1}^{\lambda_{i}-i+j} h_{p}(X, \bar{X})\right)_{1 \leqslant i, j \leqslant n}
\end{align*}
$$

Applying $C_{j}-C_{j-1}$ for $2 \leqslant j \leqslant n$, we get the required expression.
Corollary 4.8. For a partition of length at most n, the odd orthogonal character $\mathrm{so}_{\lambda}(-X)$ is given by

$$
\operatorname{so}_{\lambda}(-X)=(-1)^{\sum_{i} \lambda_{i}} \operatorname{det}\left(h_{\beta_{i}(\lambda)-n+j}(X, \bar{X})-h_{\beta_{i}(\lambda)-n-j+1}\right)_{1 \leqslant i, j \leqslant n}(X, \bar{X}) .
$$

Lemma 4.9. Let $\lambda$ be a partition of length at most tn and $\beta_{i}=t n-i$. Fix $0 \leqslant z \leqslant t-1$ and $0 \leqslant y \leqslant\left\lfloor\frac{t-z-1}{2}\right\rfloor$. Assume $\sum_{i=y}^{t-1-z-y} n_{i}=(t-z-2 y) n$. Define $a(t-z-2 y) n \times(t-z-2 y) n$
matrix with $(t-z-2 y) \times(t-z-2 y)$ blocks as

$$
\Pi^{ \pm}:=\left(\left(H_{\beta_{i}^{(p)}(\lambda)-\beta_{j t-q}^{\prime}}^{\prime} \pm H_{\beta_{i}^{(p)}(\lambda)+\beta_{j t-q}-2 t n+z+1}^{\prime}\right)_{\substack{1 \leqslant i \leqslant n_{p} \\ 1 \leqslant j \leqslant n}}\right)_{y \leqslant p, q \leqslant t-z-1-y}
$$

where $H_{m}^{\prime}=h_{m}\left(X, \omega X, \ldots, \omega^{t-1} X, \bar{X}, \omega \bar{X}, \ldots, \omega^{t-1} \bar{X}\right)$.

1. If $n_{p}+n_{t-z-1-p} \neq 2 n$ for some $p \in\left[y,\left\lfloor\frac{t-z-1}{2}\right\rfloor\right]$, then $\operatorname{det} \Pi=0$.
2. If $n_{p}+n_{t-z-1-p}=2 n$ for all $p \in\left[y,\left\lfloor\frac{t-z-1}{2}\right]\right]$, then

$$
\operatorname{det} \Pi^{+}=(-1)^{\epsilon^{+}} \prod_{i=y}^{\left\lfloor\frac{t-z-2}{2}\right\rfloor} s_{\mu_{i}^{(z)}}\left(X^{t}, \bar{X}^{t}\right) \times \begin{cases}\text { so } \left._{\lambda}{ }_{\lambda} \frac{t-z-1}{2}\right)\left(X^{t}\right) & t-z \text { odd }, \\ 1 & t-z \text { even },\end{cases}
$$

and

$$
\operatorname{det} \Pi^{-}=(-1)^{\epsilon^{-}} \prod_{i=y}^{\left\lfloor\frac{t-z-2}{2}\right\rfloor} s_{\mu_{i}^{(z)}}\left(X^{t}, \bar{X}^{t}\right) \times \begin{cases}(-1)^{\sum_{i} \lambda_{i}^{(t-z-1 / 2)}} \text { So }_{\lambda}\left(\frac{t-z-1}{2}\right)\left(-X^{t}\right) & t-z \text { odd }, \\ 1 & t-z \text { even },\end{cases}
$$

where

$$
\epsilon^{ \pm}=\sum_{i=\left\lfloor\frac{t-z+1}{2}\right\rfloor}^{t-z-y-1}\left(\frac{n_{i}(\lambda)\left(n_{i}(\lambda) \pm 1\right)}{2}-\frac{n(n \pm 1)}{2}\right)+ \begin{cases}n \sum_{i=\frac{t-z+1}{2}}^{t-z-y-1}\left(n_{i}-n\right) & t-z \text { odd } \\ 0 & t-z \text { even }\end{cases}
$$

and $\mu_{i}^{(z)}=\lambda_{1}^{(t-z-1-i)}+\left(\lambda^{(i)}, 0, \ldots, 0,-\operatorname{rev}\left(\lambda^{(t-z-1-i)}\right)\right)$ has $2 n$ parts.

Proof. Consider the permutations $\zeta$ and $\eta$ in $S_{(t-z-2 y) n}$ which rearranges the columns and rows of $\Pi^{ \pm}$respectively, blockwise in the following order: $1, t-z-2 y, 2, t-z-2 y-1, \ldots$. In other words, $\zeta$ and $\eta$ can be written in one line notation as

$$
\begin{aligned}
& \zeta=(\underbrace{1, \ldots, n_{y}}_{n_{y}}, \underbrace{(t-z-2 y) n-n_{t-z-y-1}+1, \ldots,(t-z-2 y)}_{n_{t-z-y-1}} n, \underbrace{n_{y}+1, \ldots, n_{y}+n_{y+1}}_{n_{y+1}}, \\
&\quad \underbrace{(t-z-2 y) n-n_{t-z-y-1}-n_{t-z-y-2}+1, \ldots,(t-z-2 y) n-n_{t-z-y-1}}_{n_{t-z-y-2}}, \ldots)
\end{aligned}
$$

$\begin{aligned} \text { and } \eta=(\underbrace{1, \ldots, n}_{n}, & \underbrace{(t-z-2 y-1) n+1, \ldots,(t-z-2 y) n}_{n}, \\ & \underbrace{n+1, \ldots, 2 n}_{n}, \underbrace{(t-z-2 y-2) n+1, \ldots,(t-z-2 y-1) n}_{n}, \ldots) .\end{aligned}$
Then, the number of inversions of $\zeta$ and $\eta$ is

$$
\begin{equation*}
\operatorname{inv}(\zeta)=\sum_{i=\left\lfloor\frac{t-z+3}{2}\right\rfloor}^{t-z-y} n_{i-1}\left((t-z-2 y) n-\left(n_{y}+\cdots+n_{t-z-i}\right)-\left(n_{i-1}+\cdots+n_{t-z-y-1}\right)\right) \tag{4.1.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{inv}(\eta)=\sum_{i=\left\lfloor\frac{t-z+3}{2}\right\rfloor}^{t-z-y} n^{2}(2 i-2-t+z) \tag{4.1.6}
\end{equation*}
$$

Then permuting the columns and rows of $\Pi$ by $\zeta$ and $\eta$ respectively and then using Theorem 2.6, it can be seen that

$$
\operatorname{det} \Pi^{ \pm}=\operatorname{sgn}(\zeta) \operatorname{sgn}(\eta)\left(\begin{array}{cccc}
W_{y}^{ \pm} & & &  \tag{4.1.7}\\
& W_{y^{+1}}^{ \pm} & & 0 \\
& & \ddots & \\
0 & & & W_{\left\lfloor\frac{t-z-1}{2}\right\rfloor}^{ \pm}
\end{array}\right)
$$

where
$W_{s}^{ \pm}=\left(\begin{array}{c|c}\left(h_{\frac{\beta_{i}^{(s)}(\lambda)-\beta_{j t-s}}{t}}(X, \bar{X})\right)_{\substack{1 \leqslant i \leqslant n_{s} \\ 1 \leqslant j \leqslant n}} & \left( \pm H_{\frac{\beta_{i}^{(s)}(\lambda)+\beta(j-1) t+s+z+1-2 t n+z+1}{t}}\right)_{\substack{1 \leqslant i \leqslant n_{s} \\ 1 \leqslant j \leqslant n}} \\ \left( \pm H_{\frac{\left.\beta_{i}^{(t-z-1-s)}(\lambda)+\beta_{j t-s}-2 t n+z+1\right)}{t}}\right)_{\substack{1 \leqslant i \leqslant n_{t-z-1-s} \\ 1 \leqslant j \leqslant n}} & \left(H_{\left(\begin{array}{l}\beta_{i}^{(t-z-1-s)}(\lambda)-\beta(j-1) t+s+z+1 \\ t\end{array}\right.}\right)_{\substack{1 \leqslant i \leqslant n_{t-z-1-s} \\ 1 \leqslant j \leqslant n}}\end{array}\right)$
and

Using Proposition 2.3 to simplify indices of complete symmetric functions in the above matrices, we have

$$
W_{s}^{ \pm}=\left(\begin{array}{c|c}
\left(H_{\beta_{i}\left(\lambda^{(s)}\right)-n+j}\right)_{\substack{1 \leqslant i \leqslant n_{s} \\
1 \leqslant j \leqslant n}} & \left( \pm H_{\beta_{i}\left(\lambda^{(s)}\right)-n-j+1}\right)_{\substack{1 \leqslant i \leqslant n_{s} \\
1 \leqslant j \leqslant n}} \\
\hline\left( \pm H_{\beta_{i}\left(\lambda^{(t-z-1-s)}\right)-n-j+1}\right)_{\substack{1 \leqslant i \leqslant n_{t-z-1-s} \\
1 \leqslant j \leqslant n}} & \left(H_{\beta_{i}\left(\lambda^{(t-z-1-s)}\right)-n+j}\right)_{\substack{1 \leqslant i \leqslant n_{t-z-1-s} \\
1 \leqslant j \leqslant n}}
\end{array}\right)
$$

and

$$
\left.\left.W_{\frac{t-z-1}{2}}^{ \pm}=\left(H_{\beta_{i}\left(\lambda\left(\frac{t-z-1}{2}\right)\right.}\right)_{-n+j} \pm H_{\beta_{i}(\lambda}\left(\frac{t-z-1}{2}\right)\right)-n-j+1\right) .
$$

If $n_{s}+n_{t-z-1-s} \neq 2 n$ for some $s \in\left[y,\left\lfloor\frac{t-z-1}{2}\right\rfloor\right]$, then $W_{s}^{ \pm}$is not a square matrix. This implies

$$
\operatorname{det} \Pi^{ \pm}=0
$$

If $n_{s}+n_{t-z-1-s}=2 n$ for all $s \in\left[y,\left\lfloor\frac{t-z-1}{2}\right\rfloor\right]$, then applying $C_{j} \leftrightarrow C_{n+j}$ for all $j \in[n]$ and then using Lemma 4.5, we see that

$$
\begin{equation*}
\operatorname{det} W_{s}^{ \pm}=\frac{(-1)^{\frac{n_{t-z-1-s}(\lambda)\left(n_{t-z-1-s}(\lambda) \pm_{1}\right)}{2}}}{(-1)^{\frac{n(n \pm 1)}{2}}} s_{\mu_{s}^{(z)}}\left(X^{t}, \bar{X}^{t}\right), \tag{4.1.8}
\end{equation*}
$$

where $\mu_{s}^{(z)}=\lambda_{1}^{(t-z-1-s)}+\left(\lambda^{(s)}, 0, \ldots, 0,-\operatorname{rev}\left(\lambda^{(t-z-1-s)}\right)\right), y \leqslant s \leqslant\left\lfloor\frac{t-z-2}{2}\right\rfloor$. Using Lemma 4.7 and, we have

$$
\begin{equation*}
\operatorname{det} W_{\frac{t-z-1}{2}}^{+}=\operatorname{so}_{\lambda}\left(\frac{t-z-1}{2}\right)\left(X^{t}\right), \quad \operatorname{det} W_{\frac{t-z-1}{2}}^{-}=(-1)^{\sum_{i} \lambda_{i}^{(t-z-1 / 2)}} \text { so }_{\lambda}\left(\frac{t-z-1}{2}\right)\left(-X^{t}\right) . \tag{4.1.9}
\end{equation*}
$$

Substituting 4.1.8 and 4.1.9) in 4.1.7, we see that

$$
\begin{aligned}
& \operatorname{det} \Pi^{ \pm}=\operatorname{sgn}(\zeta) \operatorname{sgn}(\eta) \prod_{i=y}^{\left\lfloor\frac{t-z-2}{2}\right\rfloor} \frac{(-1)^{\frac{n_{t-z-1-i}(\lambda)\left(n_{t-z-1-i}(\lambda) \pm 1\right)}{2}}}{(-1)^{\frac{n(n+1)}{2}}} s_{\mu_{i}^{(z)}}\left(X^{t}, \bar{X}^{t}\right) \\
& \times \begin{cases}\operatorname{det} W_{\frac{t-z-1}{2}}^{ \pm} & t-z \text { odd } \\
1 & t-z \text { even }\end{cases}
\end{aligned}
$$

All that remains is to compute the sign. By 4.1.5) and 4.1.6, we get

$$
\operatorname{inv}(\zeta)-\operatorname{inv}(\eta)=\sum_{i=\left\lfloor\frac{t-z+1}{2}\right\rfloor}^{t-z-y-1}\left(n_{i}-n\right)(2 i-2-t+z) n
$$

Therefore, if $t-z$ is even, then $\operatorname{inv}(\zeta)-\operatorname{inv}(\eta)$ is even and $\operatorname{sgn}(\zeta) \operatorname{sgn}(\eta)$ is 1 . If $t-z$ is odd, then the only contribution for $\operatorname{sgn}(\zeta) \operatorname{sgn}(\eta)$ comes from $n \sum_{i=\frac{t-z+1}{2}}^{t-z-y-1}\left(n_{i}-n\right)$, since other terms are even. Substituting the terms completes the proof.

### 4.2 Factorization of other classical characters

Proof of Theorem 3.5. Using the Jacobi-Trudi identity 2.4.10 for the symplectic characters, we see that the desired symplectic character is

$$
\begin{aligned}
& \operatorname{sp}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right) \\
& \quad=\operatorname{det}\left(\begin{array}{llll}
H_{\lambda_{i}-i+1}^{\prime} & H_{\lambda_{i}-i+2}^{\prime}+H_{\lambda_{i}-i}^{\prime} & \ldots & H_{\lambda_{i}-i+t n}^{\prime}+H_{\lambda_{i}-i-t n+2}^{\prime}
\end{array}\right)_{1 \leqslant i \leqslant t n} \\
& \quad=\operatorname{det}\left(\begin{array}{llll}
H_{\beta_{i}(\lambda)-\beta_{1}}^{\prime} & H_{\beta_{i}(\lambda)-\beta_{2}}^{\prime}+H_{\beta_{i}(\lambda)+\beta_{2}-2 t n+2}^{\prime} & \ldots & H_{\beta_{i}(\lambda)-\beta_{t n}}^{\prime}+H_{\beta_{i}(\lambda)+\beta_{t n}-2 t n+2}^{\prime}
\end{array}\right)_{1 \leqslant i \leqslant t n}
\end{aligned}
$$

where $H_{m}^{\prime}=h_{m}\left(X, \omega X, \ldots, \omega^{t-1} X, \bar{X}, \omega \bar{X}, \ldots, \omega^{t-1} \bar{X}\right)$ and $\beta_{i}=t n-i$. Permuting the rows and the columns of the determinant by $\sigma_{\lambda}$ and $\sigma_{\varnothing}$ from 6.1.3) and 6.1.4, respectively and then using (2.2.9), we see that the symplectic character is

$$
\operatorname{sgn}\left(\sigma_{\lambda}\right) \operatorname{sgn}\left(\sigma_{\varnothing}\right) \operatorname{det}\left(\begin{array}{c|c}
P_{(t-1) n \times \sum_{i=0}^{t-2} n_{i}(\lambda)} & 0  \tag{4.2.1}\\
\hline 0 & Q_{n \times n_{t-1}(\lambda)}
\end{array}\right)
$$

where

$$
P=\left(\left(H_{\beta_{i}^{(p)}(\lambda)-\beta_{j t-q}}^{\prime}+H_{\beta_{i}^{(p)}(\lambda)+\beta_{j t-q}-2 t n+2}^{\prime}\right)_{\substack{1 \leqslant i \leqslant n_{p} \\ 1 \leqslant j \leqslant n}}\right)_{0 \leqslant p, q \leqslant t-2}
$$

and

$$
\begin{equation*}
Q=\left(H_{\beta_{i}^{(t-1)}(\lambda)-\beta_{1}}^{\prime} \mid\left(H_{\beta_{i}^{(t-1)}(\lambda)-\beta_{(j-1) t+1}^{\prime}}^{\prime}+H_{\beta_{i}^{(t-1)}(\lambda)+\beta_{(j-1) t+1}^{\prime}-2 t n+2}^{\prime}\right)_{2 \leqslant j \leqslant n}\right)_{1 \leqslant i \leqslant n_{t-1}} \tag{4.2.2}
\end{equation*}
$$

If $\operatorname{core}_{t}(\lambda)$ is not a symplectic $t$-core, then by Corollary 3.22, either $n_{t-1}(\lambda) \neq n$ or $n_{i}(\lambda)+n_{t-2-i}(\lambda) \neq 2 n$ for some $i \in\left[0,\left\lfloor\frac{t-2}{2}\right\rfloor\right]$. In the first case, i.e. if $n_{t-1}(\lambda) \neq n$, then by (4.2.1),

$$
\operatorname{sp}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=0
$$

In the second case, if $n_{i}+n_{t-2-i} \neq 2 n$, for some $i \in\left[0,\left\lfloor\frac{t-2}{2}\right\rfloor\right]$, then using Lemma 4.9 for $y=0$ and $z=1$, $\operatorname{det} P$ is zero. Hence,

$$
\operatorname{sp}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=0
$$

Now suppose $n_{i}+n_{t-2-i}=2 n, \forall i \in\left[0,\left\lfloor\frac{t-2}{2}\right\rfloor\right]$. Then using Lemma 4.9 for $y=0$ and $z=1, \operatorname{det} P$ is

$$
\operatorname{det} P=(-1)^{\epsilon^{\prime}} \prod_{i=0}^{\left\lfloor\frac{t-3}{2}\right\rfloor} s_{\mu_{i}^{(z)}}\left(X^{t}, \bar{X}^{t}\right) \times \begin{cases}\text { so } \left._{\lambda}{ }_{\lambda} \frac{t-2}{2}\right)\left(X^{t}\right) & t \text { even },  \tag{4.2.3}\\ 1 & t \text { odd },\end{cases}
$$

where

$$
\epsilon^{\prime}=\sum_{i=\left\lfloor\frac{t}{2}\right\rfloor}^{t-2}\left(\frac{n_{i}(\lambda)\left(n_{i}(\lambda)+1\right)}{2}-\frac{n(n+1)}{2}\right)+ \begin{cases}n \sum_{i=\frac{t}{2}}^{t-2}\left(n_{i}-n\right) & t \text { even } \\ 0 & t \text { odd }\end{cases}
$$

and $\mu_{i}^{(1)}=\lambda_{1}^{(t-2-i)}+\left(\lambda^{(i)}, 0, \ldots, 0,-\operatorname{rev}\left(\lambda^{(t-2-i)}\right)\right)$ has $2 n$ parts for $0 \leqslant i \leqslant\left\lfloor\frac{t-3}{2}\right\rfloor$. Using Proposition 2.3, we have

$$
\begin{equation*}
\lambda_{i}^{(t-1)}-i+j=\frac{1}{t}\left(\beta_{i}^{(t-1)}(\lambda)-\beta_{(j-1) t+1}\right) \tag{4.2.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\lambda_{i}^{(t-1)}-i-j=\frac{1}{t}\left(\beta_{i}^{(t-1)}(\lambda)-\beta_{(j-1) t+1}-2 t(j-1)\right) . \tag{4.2.5}
\end{equation*}
$$

Using (2.2.9) and then Substituting (4.2.4) and (4.2.5) in 4.2.2), we see that the determinant of $Q$ is

$$
\begin{align*}
& \operatorname{det} Q \\
& =\operatorname{det}\left(h_{\lambda_{i}^{(t-1)}-i+1}(X, \bar{X}) \mid\left(h_{\lambda_{i}^{(t-1)}-i+j}(X, \bar{X})+h_{\lambda_{i}^{(t-1)}-i-j+2}(X, \bar{X})\right)_{2 \leqslant j \leqslant n}\right)  \tag{4.2.6}\\
& \quad=\operatorname{sp}_{\lambda^{(t-1)}}\left(X^{t}\right) .
\end{align*}
$$

Substituting (4.2.3) and (4.2.6) in (4.2.1) gives the desired symplectic character (3.1.6) and
$\epsilon=\frac{t(t-1)}{2} \frac{n(n+1)}{2}+\sum_{i=\left\lfloor\frac{t}{2}\right\rfloor}^{t-2}\left(\frac{n_{i}(\lambda)\left(n_{i}(\lambda)+1\right)}{2}-\frac{n(n+1)}{2}\right)+ \begin{cases}n \sum_{i=\frac{t}{2}}^{t-2}\left(n_{i}-n\right) & t \text { even, } \\ 0 & t \text { odd },\end{cases}$

Now to compute the sign, we simplify the expression for $\epsilon$. Since $\frac{(t-1)(t+1)}{2} \frac{n(n+1)}{2}$ is even for odd $t$ and the parity of $\frac{n(n+1)}{2} \frac{\left(t^{2}-2 t+2\right)}{2}$ is the same as the parity of $\frac{n(n+1)}{2}$ for even $t$, $\epsilon$ has the same parity as

$$
-\sum_{i=\left\lfloor\frac{t}{2}\right\rfloor}^{t-2}\binom{n_{i}(\lambda)+1}{2}+ \begin{cases}\frac{n(n+1)}{2}+n r & t \text { even } \\ 0 & t \text { odd }\end{cases}
$$

where $r$ is the rank from (3.2.6). This completes the proof.

For the even orthogonal characters, we have $G=\mathrm{O}_{2 t n}$, the orthogonal group of $(2 t n) \times(2 t n)$ square matrices.

Proof of Theorem 3.9. Using the Jacobi-Trudi identity (2.4.11) for the even orthogonal characters, the required even orthogonal character is

$$
\begin{align*}
\mathrm{o}_{\lambda}^{\text {even }}\left(X, \omega X, \ldots, \omega^{t-1} X\right) & =\operatorname{det}\left(H_{\lambda_{i}-i+j}^{\prime}-H_{\lambda_{i}-i-j}^{\prime}\right)_{1 \leqslant i, j \leqslant t n} \\
& =\operatorname{det}\left(H_{\beta_{i}(\lambda)-\beta_{j}}^{\prime}-H_{\beta_{i}(\lambda)+\beta_{j}-2 t n}^{\prime}\right)_{1 \leqslant i, j \leqslant t n}, \tag{4.2.7}
\end{align*}
$$

where $H_{m}^{\prime}=h_{m}\left(X, \omega X, \ldots, \omega^{t-1} X, \bar{X}, \omega \bar{X}, \ldots, \omega^{t-1} \bar{X}\right)$ and $\beta_{j}=t n-j$. Permuting the rows and columns of the determinant by $\sigma_{\lambda}$ and $\sigma_{\varnothing}$ from (6.1.3) and (6.1.4) respectively and then using $(2.2 .9)$, we see that the even orthogonal character is

$$
\operatorname{sgn}\left(\sigma_{\lambda}\right) \operatorname{sgn}\left(\sigma_{\varnothing}\right) \operatorname{det}\left(\begin{array}{c|c}
P_{1 \leqslant i \leqslant n_{0}(\lambda)}^{1 \leqslant j \leqslant n} & 0 \\
\hline 0 & Q_{(t-1) n \times \sum_{i=1}^{t-1} n_{i}(\lambda)}
\end{array}\right),
$$

where

$$
P=\left(H_{\beta_{i}^{(0)}(\lambda)-\beta_{t j}}^{\prime}-H_{\beta_{i}^{(0)}(\lambda)+\beta_{t j}-2 t n}^{\prime}\right)_{\substack{1 \leqslant i \leqslant n_{0}(\lambda) \\ 1 \leqslant j \leqslant n}}
$$

and

$$
\left.Q=\left(\left(H_{\beta_{i}^{(p)}(\lambda)-\beta_{t j-q}}^{\prime}-H_{\beta_{i}^{(p)}(\lambda)+\beta_{t j-q}-2 t n}^{\prime}\right)\right)_{\substack{1 \leqslant i \leqslant n_{p}(\lambda) \\ 1 \leqslant j \leqslant n}}\right)_{1 \leqslant p, q \leqslant t-1} .
$$

If $\operatorname{core}_{t}(\lambda)$ is not an orthogonal $t$-core, then by Corollary 3.23, either $n_{0}(\lambda) \neq n$ or $n_{i}(\lambda)+n_{t-i}(\lambda) \neq 2 n$ for some $i \in\left\{1, \ldots,\left\lfloor\frac{t}{2}\right\rfloor\right\}$. In the first case, i.e. if $n_{0}(\lambda) \neq n$, then

$$
\mathrm{o}_{\lambda}^{\text {even }}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=0 .
$$

In the second case, if $n_{i}+n_{t-i} \neq 2 n$, for some $i \in\left[\left\lfloor\frac{t}{2}\right\rfloor\right]$, then using Lemma 4.9 for $y=1$
and $z=-1$, $\operatorname{det} Q$ is zero. Hence,

$$
\mathrm{o}_{\lambda}^{\text {even }}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=0
$$

If $\operatorname{core}_{t}(\lambda)$ is not an orthogonal $t$-core, then by Corollary 3.23, $n_{i}+n_{t-i}=2 n, \forall i \in\left[\left\lfloor\frac{t}{2}\right]\right]$. Then using Lemma 4.9 for $y=1$ and $z=-1$, $\operatorname{det} Q$ is

$$
\operatorname{det} Q=(-1)^{\epsilon} \prod_{i=1}^{\left\lfloor\frac{t-1}{2}\right\rfloor} s_{\mu_{i}^{(2)}}\left(X^{t}, \bar{X}^{t}\right) \times \begin{cases}(-1)^{\sum_{i} \lambda_{i}^{(t / 2)}} \text { so }_{\lambda}\left(\frac{t}{2}\right)\left(-X^{t}\right) & t \text { even, } \\ 1 & t \text { odd },\end{cases}
$$

where

$$
\epsilon=\sum_{i=\left\lfloor\frac{t+2}{2}\right\rfloor}^{t-1}\left(\frac{n_{i}(\lambda)\left(n_{i}(\lambda)-1\right)}{2}-\frac{n(n-1)}{2}\right)+ \begin{cases}n \sum_{i=\frac{t+2}{2}}^{t-1}\left(n_{i}-n\right) & t \text { even } \\ 0 & t \text { odd }\end{cases}
$$

and $\mu_{i}^{(2)}=\lambda_{1}^{(t-i)}+\left(\lambda^{(i)}, 0, \ldots, 0,-\operatorname{rev}\left(\lambda^{(t-i)}\right)\right)$ has $2 n$ parts for $1 \leqslant i \leqslant\left\lfloor\frac{t-1}{2}\right\rfloor$. Using Proposition 2.3, we have

$$
\begin{equation*}
\lambda_{i}^{(0)}-i+j=\lambda_{i}^{(0)}+n-i-n+j=\frac{1}{t}\left(\beta_{i}^{(0)}\right)-n+j=\frac{1}{t}\left(\beta_{i}^{(0)}-t n+t j\right)=\frac{1}{t}\left(\beta_{i}^{(t-1)}-\beta_{t j}\right), \tag{4.2.8}
\end{equation*}
$$

So, the $\operatorname{det} P=\mathrm{o}_{\lambda^{(0)}}^{\text {even }}\left(X^{t}\right)$. Finally, the even orthogonal character is given by

$$
(-1)^{\epsilon^{\prime}} \operatorname{sgn}\left(\sigma_{\lambda}\right) \mathrm{o}_{\lambda(0)}^{\text {even }}\left(X^{t}\right) \prod_{i=1}^{\left\lfloor\frac{t-1}{2}\right\rfloor} s_{\mu_{i}^{(2)}}\left(X^{t}, \bar{X}^{t}\right) \times \begin{cases}(-1)^{\sum_{i=1}^{n} \lambda_{i}^{(t / 2)}} \mathrm{so}_{\lambda^{(t / 2)}}\left(-X^{t}\right) & t \text { even }, \\ 1 & t \text { odd }\end{cases}
$$

where

$$
\epsilon^{\prime}=\frac{t(t-1)}{2} \frac{n(n+1)}{2}+\sum_{i=\left\lfloor\frac{t+2}{2}\right\rfloor}^{t-1}\left(\frac{n_{i}(\lambda)\left(n_{i}(\lambda)-1\right)}{2}-\frac{n(n-1)}{2}\right)+ \begin{cases}n \sum_{i=\frac{t+2}{2}}^{t-1}\left(n_{i}-n\right) & t \text { even }, \\ 0 & t \text { odd }\end{cases}
$$

After simplifications, the parity of $\epsilon^{\prime}$ shown to be the same as

$$
\epsilon=-\sum_{i=\left\lfloor\frac{t+2}{2}\right\rfloor}^{t-1}\binom{n_{i}(\lambda)}{2}+ \begin{cases}\frac{n(n+t-1)}{2}+n r & t \text { even } \\ \frac{t-1) n}{2} & t \text { odd }\end{cases}
$$

This completes the proof.
For the odd orthogonal characters, we have $G=\mathrm{SO}_{2 t n+1}$, the orthogonal group of $(2 t n+1) \times(2 t n+1)$ square matrices.

Proof of Theorem 3.11. Using Lemma 4.7for the odd orthogonal characters, the desired odd orthogonal character is

$$
\begin{equation*}
\operatorname{so}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=\operatorname{det}\left(H_{\beta_{i}(\lambda)-\beta_{j}}^{\prime}+H_{\beta_{i}(\lambda)+\beta_{j}-2 t n+1}^{\prime}\right)_{1 \leqslant i, j \leqslant t n} \tag{4.2.9}
\end{equation*}
$$

where $H_{m}^{\prime}=h_{m}\left(X, \omega X, \ldots, \omega^{t-1} X, \bar{X}, \omega \bar{X}, \ldots, \omega^{t-1} \bar{X}\right)$. Permuting the rows and columns by $\sigma_{\lambda}$ and $\sigma_{\varnothing}$ from (6.1.3) and (6.1.4 respectively, we see that the odd orthogonal character is

$$
\operatorname{sgn}\left(\sigma_{\lambda}\right) \operatorname{sgn}\left(\sigma_{\varnothing}\right) \operatorname{det}\left(\left(H_{\beta_{i}^{(p)}(\lambda)-\beta_{t j-q}}^{\prime}+H_{\beta_{i}^{(p)}(\lambda)+\beta_{t j-q}-2 t n+1}^{\prime}\right)_{\substack{1 \leqslant i \leqslant n_{p} \\ 1 \leqslant j \leqslant n}}\right)_{0 \leqslant p, q \leqslant t-1}
$$

If $\operatorname{core}_{t}(\lambda)$ is not self-conjugate, then using Corollary 3.18, $n_{p}+n_{t-1-p} \neq 2 n$ for some $p \in\left[0,\left\lfloor\frac{t-1}{2}\right\rfloor\right]$. Applying Lemma 4.9 for $z=y=0$, we get

$$
\operatorname{so}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X\right)=0
$$

If $\operatorname{core}_{t}(\lambda)$ is self-conjugate with rank $r$, then using Corollary 3.18, $n_{p}+n_{t-1-p} \neq 2 n$ for all $p \in\left[0,\left\lfloor\frac{t-1}{2}\right\rfloor\right]$. Applying Lemma 4.9 for $z=y=0$

$$
(-1)^{\epsilon} \operatorname{sgn}\left(\sigma_{\lambda}\right) \operatorname{sgn}\left(\sigma_{\varnothing}\right) \prod_{i=0}^{\left\lfloor\frac{t-2}{2}\right\rfloor} s_{\mu_{i}^{(3)}}\left(X^{t}, \bar{X}^{t}\right) \times \begin{cases}\operatorname{so}_{\lambda}\left(\frac{t-1}{2}\right) & \left(X^{t}\right) \\ 1 & t \text { odd } \\ 1 & t \text { even }\end{cases}
$$

where

$$
\epsilon=\sum_{i=\left\lfloor\frac{t+1}{2}\right\rfloor}^{t-1}\left(\frac{n_{i}(\lambda)\left(n_{i}(\lambda)+1\right)}{2}-\frac{n(n+1)}{2}\right)+ \begin{cases}n \sum_{i=\frac{t+1}{2}}^{t-1}\left(n_{i}-n\right) & t \text { odd } \\ 0 & t \text { even }\end{cases}
$$

and $\mu_{i}^{(3)}=\lambda_{1}^{(t-1-i)}+\left(\lambda^{(i)}, 0, \ldots, 0,-\operatorname{rev}\left(\lambda^{(t-1-i)}\right)\right)$ has $2 n$ parts for $0 \leqslant i \leqslant\left\lfloor\frac{t-2}{2}\right\rfloor$.
$\epsilon^{\prime}=\frac{t(t-1)}{2} \frac{n(n+1)}{2}-\sum_{i=\left\lfloor\frac{t+1}{2}\right\rfloor}^{t-1}\left(\frac{n_{i}(\lambda)\left(n_{i}(\lambda)+1\right)}{2}-\frac{n(n+1)}{2}\right)+ \begin{cases}n \sum_{i=\frac{t+1}{2}}^{t-1}\left(n_{i}-n\right) & t \text { odd }, \\ 0 & t \text { even },\end{cases}$

After similar simplifications, $\epsilon^{\prime}$ turns out to have the same parity as

$$
-\sum_{i=\left\lfloor\frac{t+1}{2}\right\rfloor}^{t-1}\binom{n_{i}(\lambda)+1}{2}+ \begin{cases}n r & t \text { odd } \\ 0 & t \text { even }\end{cases}
$$

where $r$ is the rank from 3.2.8). This completes the proof.

## Chapter 5

## Factorization of classical characters twisted by roots of unity: II

In this chapter, we extend the results to the groups $\mathrm{GL}_{t n+m}(0 \leqslant m \leqslant t-1), \mathrm{SO}_{2 t n+3}$, $\mathrm{Sp}_{2 t n+2}$ and $\mathrm{O}_{2 t n+2}$ evaluated at similar specializations: (1) for the $\mathrm{GL}_{t n+m}(\mathbb{C})$ case, we set the first $t n$ elements to $\omega^{j} x_{i}$ for $0 \leqslant j \leqslant t-1$ and $1 \leqslant i \leqslant n$ and the remaining $m$ to $y, \omega y, \ldots, \omega^{m-1} y$; (2) for the other three families, the same specializations but with $m=1$. Our motivation is the conjectures of Wagh and Prasad 91 relating the irreducible representations of $\operatorname{Spin}_{2 n+1}$ and $\mathrm{SL}_{2 n}, \mathrm{SL}_{2 n+1}$ and $\mathrm{Sp}_{2 n}$ as well as $\mathrm{Spin}_{2 n+2}$ and $\mathrm{Sp}_{2 n}$. In each case, we characterize partitions for which the character value is nonzero in terms of what we call $\left(z_{1}, z_{2}, k\right)$-asymmetric partitions, where $z_{1}, z_{2}$ and $k$ are integers which depend on the group. We give statements of results and illustrative examples in Section 5.1. We formulate results on beta sets, generating functions and determinant identities in Section 5.2. We prove the Schur factorization result in Section 5.3. We prove the new factorizations of other classical characters in Section 5.4. Finally, we prove generating function formulas for $\left(z_{1}, z_{2}, k\right)$-asymmetric partitions and $\left(z_{1}, z_{2}, k\right)$ asymmetric $t$-cores in Section 5.5. A preprint of this work has appeared on arXiv [70].

### 5.1 Main results

Recall, $X=\left(x_{1}, \ldots, x_{n}\right)$ and $\omega$ is a primitive $t$ 'th root of unity. Fix $0 \leqslant m \leqslant t-1$. We first consider the specialized Schur polynomial evaluated at elements twisted by the $t^{\prime}$ th roots of unity. We denote the indeterminates by $X, \omega X, \omega^{2} X, \ldots, \omega^{t-1} X, y, \ldots, \omega^{m-1} y$.

Let $E=\left(e_{1}, \ldots, e_{m}\right)$ such that $t-1 \geqslant e_{1}>\cdots>e_{m} \geqslant 0$. We extend $E$ for enumerating the set $\{0, \ldots, t-1\} \backslash\left\{e_{i}\right\}_{i \in[m]}$ as $\left\{e_{m+1}<\cdots<e_{t}\right\}$, denoted $\bar{E}$, for convenience. For a partition $\lambda$ of length at most $t n+m$, let $\sigma_{\lambda}^{E}$ be the permutation in $S_{t n+m}$ such that
it rearranges parts of $\beta(\lambda)$ in the following way:

$$
\begin{equation*}
\beta_{\sigma_{\lambda}^{E}(j)}(\lambda) \equiv e_{q} \quad(\bmod t), \quad \text { for } \quad \sum_{i=1}^{q-1} n_{e_{i}}(\lambda)+1 \leqslant j \leqslant \sum_{i=1}^{q} n_{e_{i}}(\lambda) \tag{5.1.1}
\end{equation*}
$$

arranged in decreasing order for each $q \in\{1, \ldots, t\}$. For simplicity, we write $\sigma_{\lambda}^{\varnothing}$ as $\sigma_{\lambda}$. For the empty partition, $\beta(\varnothing, t n+1)=(t n, t n-1, t n-2, \ldots, 0)$ with

$$
n_{q}(\varnothing, t n+1)= \begin{cases}n+1 & q=0 \\ n & 1 \leqslant q \leqslant t-1\end{cases}
$$

and

$$
\begin{equation*}
\sigma_{\varnothing}=(1, t+1, \ldots, n t+1, t, \ldots, n t, \ldots, 2, \ldots,(n-1) t+2) \tag{5.1.2}
\end{equation*}
$$

in one-line notation with $\operatorname{sgn}\left(\sigma_{\varnothing}\right)=(-1)^{\frac{t(t-1)}{2} \frac{n(n+1)}{2}}$.
Theorem 5.1. Fix $0 \leqslant m \leqslant t-1$. Let $\lambda$ be a partition of length at most $t n+m$ indexing an irreducible representation of $\mathrm{GL}_{t n+m}$ and quo $_{t}(\lambda)=\left(\lambda^{(0)}, \ldots, \lambda^{(t-1)}\right)$. Then the $\mathrm{GL}_{t n+m}$-character $s_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X, y, \omega y, \ldots, \omega^{m-1} y\right)$ is as follows:

1. If $\operatorname{core}_{t}(\lambda)=\nu:=\left(\nu_{1}, \ldots, \nu_{m}\right)$ for some $\nu_{1} \leqslant t-m$, then

$$
\begin{align*}
s_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X, y, \omega y, \ldots, \omega^{m-1} y\right) & =\operatorname{sgn}\left(\sigma_{\lambda}^{\beta(\nu)}\right) \operatorname{sgn}\left(\sigma_{\varnothing}^{\beta(\nu)}\right) \\
& \times s_{\nu}\left(1, \omega, \ldots, \omega^{m-1}\right) \prod_{i=1}^{m} s_{\lambda^{\left(\beta_{i}(\nu)\right)}}\left(X^{t}, y^{t}\right) \prod_{\substack{j=0 \\
j \notin \beta(\nu)}}^{t-1} s_{\lambda(j)}\left(X^{t}\right) . \tag{5.1.3}
\end{align*}
$$

2. Otherwise,

$$
\begin{equation*}
s_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X, y, \omega y, \ldots, \omega^{m-1} y\right)=0 \tag{5.1.4}
\end{equation*}
$$

In other words, the nonzero $\mathrm{GL}_{t n+m}$ character is the product of $m \mathrm{GL}_{n+1}$ and $(t-m)$ $\mathrm{GL}_{n}$ characters. For $m=0$ and $m=1$, Theorem 5.1 is proved by Littlewood [74, Equation $(7.3 ; 3)$ ], [74, Chapter VII, Section IX] and independently by Prasad [90, Theorem 2], [76, Theorem 4.5] for $t=2$. For $m=0$ the result is also proven in Chapter 3. In the case when $X=(1, \ldots, 1)$ and $\operatorname{core}_{t}(\lambda)$ is empty, (5.1.3) is proved in [87].

Example 5.2. For $t=2, m=1$, Theorem 5.1 says that the character of the group $\mathrm{GL}_{3}$ of the representation indexed by the partition $(a, b, c), a \geqslant b \geqslant c \geqslant 0$, evaluated at $(x,-x, y)$ is non-zero if and only if $a$ and $b$ have the same parity or $a$ and $c$ have the
opposite parity. If $\operatorname{core}_{2}(a, b, c)$ is empty, then

$$
s_{(a, b, c)}(x,-x, y)= \begin{cases}-s_{\left(\frac{a}{2}, \frac{b+1}{2}\right)}\left(x^{2}, y^{2}\right) s_{\left(\frac{c-1}{2}\right)}\left(x^{2}\right) & \text { a even, } b \text { and } c \text { odd } \\ s_{\left(\frac{a}{2}, \frac{c}{2}\right)}\left(x^{2}, y^{2}\right) s_{\left(\frac{b}{2}\right)}\left(x^{2}\right) & \text { a, b, c even } \\ -s_{\left(\frac{b-1}{2}, \frac{c}{2}\right)}\left(x^{2}, y^{2}\right) s_{\left(\frac{a+1}{2}\right)}\left(x^{2}\right) & \text { a and } b \text { odd, c even }\end{cases}
$$

and if $\operatorname{core}_{2}(a, b, c)=(1)$, then

$$
s_{(a, b, c)}(x,-x, y)= \begin{cases}y s_{\left(\frac{a-1}{2}, \frac{b}{2}\right)}\left(x^{2}, y^{2}\right) s_{\left(\frac{c}{2}\right)}\left(x^{2}\right) & \text { a odd, } b \text { and } c \text { even } \\ -y s_{\left(\frac{a-1}{2}, \frac{c-1}{2}\right)}\left(x^{2}, y^{2}\right) s_{\left(\frac{b+1}{2}\right)}\left(x^{2}\right) & a, b, c \text { odd } \\ y s_{\left(\frac{b-2}{2}, \frac{c-1}{2}\right)}\left(x^{2}, y^{2}\right) s_{\left(\frac{a+2}{2}\right)}\left(x^{2}\right) & \text { a and } b \text { even, } c \text { odd }\end{cases}
$$

We now generalize Theorem 5.1 to other classical characters for $m=1$. We first need some definitions.

Definition 5.3. Suppose $z_{1}>z_{2} \geqslant 0$ and $\lambda$ is a partition of rank $r$. We say $\lambda$ is $\left(z_{1}, z_{2}, k\right)-$ asymmetric for some $0 \leqslant k \leqslant r$, if $\lambda=\left(\alpha_{1}, \ldots, \alpha_{k}, \ldots, \alpha_{r} \mid \alpha_{1}+z_{1}, \ldots, \widehat{\alpha_{k}+z_{1}}, \ldots, \alpha_{r}+\right.$ $z_{1}, z_{2}$ ), in Frobenius coordinates for some strict partition $\alpha$, where a hat on a coordinate denotes its omission. (Here $k=0$ means no part is omitted and therefore no part is added). If in addition a $\left(z_{1}, z_{2}, k\right)$-asymmetric partition is also a $t$-core, we call it a $\left(z_{1}, z_{2}, k\right)$-asymmetric $t$-core. We denote the set of $\left(z_{1}, z_{2}, k\right)$-asymmetric partitions and $\left(z_{1}, z_{2}, k\right)$-asymmetric $t$-cores by $\mathcal{Q}_{z_{1}, z_{2}, k}$ and $\mathcal{Q}_{z_{1}, z_{2}, k}^{(t)}$ respectively.

Note that the $(z, 0,0)$-asymmetric partition is the $z$-asymmetric partition defined in Chapter 3. Recall that a partition $\lambda$ is $z$-asymmetric if $\lambda=(\alpha \mid \beta)$ where $\beta_{i}=\alpha_{i}+z$ for $1 \leqslant i \leqslant \operatorname{rk}(\lambda)$.

To state our results, define, for $\lambda=\left(\lambda_{1}, \ldots, \lambda_{n}\right)$, the reverse of $\lambda$ as $\operatorname{rev}(\lambda)=$ $\left(\lambda_{n}, \ldots, \lambda_{1}\right)$. Moreover, if $\mu=\left(\mu_{1}, \ldots, \mu_{j}\right)$ is a partition such that $\mu_{1} \leqslant \lambda_{n}$, then we write the concatenated partition $(\lambda, \mu)=\left(\lambda_{1}, \ldots, \lambda_{n}, \mu_{1}, \ldots, \mu_{j}\right)$.

For the odd orthogonal case, we take $G=\mathrm{SO}_{2 t n+3}$, the orthogonal group of ( $2 t n+$ $3) \times(2 t n+3)$ square matrices. For a partition $\lambda$, if $\operatorname{core}_{t}(\lambda)$ is either $(2,0, k)$ - or $(2,1, k)$ asymmetric for some $k \in\left[\operatorname{rk}\left(\operatorname{core}_{t}(\lambda)\right)\right]$, then by Corollary 5.20, there exists a unique $i_{0} \in\left[0,\left\lfloor\frac{t-1}{2}\right\rfloor\right]$ such that

$$
n_{i}(\lambda)+n_{t-1-i}(\lambda)=\left\{\begin{array}{ll}
2 n+1+\delta_{i_{0}, \frac{t-1}{2}} & \text { if } i=i_{0}, \\
2 n & \text { otherwise },
\end{array} \quad 0 \leqslant i \leqslant\left\lfloor\frac{t-1}{2}\right\rfloor\right.
$$

For such a partition $\lambda$, let

$$
\begin{equation*}
\epsilon_{1}(\lambda):=\left(\sum_{i=t-i_{0}}^{t-1} n_{i}(\lambda)\right)+\sum_{i=\left\lfloor\frac{t+1}{2}\right\rfloor}^{t-1}\left(\binom{n_{i}(\lambda)+1}{2}+\operatorname{tn}\left(n_{i}(\lambda)-n\right)\right), \tag{5.1.5}
\end{equation*}
$$

and $\pi_{i}^{(1)}:=\lambda_{1}^{(t-1-i)}+\left(\lambda^{(i)}, 0, \ldots, 0,-\operatorname{rev}\left(\lambda^{(t-1-i)}\right)\right)$ has $2 n+\delta_{i, i_{0}}$ parts for $0 \leqslant i \leqslant\left\lfloor\frac{t-1}{2}\right\rfloor$. We note that the empty partition is vacuously ( $2,0,0$ )-asymmetric with $i_{0}=0$. Our result for the factorization of odd orthogonal characters is as follows.

Theorem 5.4. Let $\lambda$ be a partition of length at most $t n+1$. Then the odd orthogonal character $\operatorname{so}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X, y\right)$ is as follows:

1. If $\operatorname{core}_{t}(\lambda)$ is either $(2,0, k)$ or $(2,1, k)$-asymmetric for some $k \in\left[\operatorname{rk}\left(\operatorname{core}_{t}(\lambda)\right)\right]$ and $i_{0}=\frac{t-1}{2}$, then

$$
\begin{align*}
& \operatorname{so}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X, y\right) \\
& \quad=(-1)^{\epsilon_{1}(\lambda)+n} \operatorname{sgn}\left(\sigma_{\lambda}\right) \operatorname{so}_{\left(\frac{t-1}{2}\right)}(y) \operatorname{so}_{\lambda}\left(\frac{t-1}{2}\right)\left(X^{t}, y^{t}\right) \times \prod_{i=0}^{\frac{t-3}{2}} s_{\pi_{i}^{(1)}}\left(X^{t}, \bar{X}^{t}\right) . \tag{5.1.6}
\end{align*}
$$

2. If $\operatorname{core}_{t}(\lambda)$ is either $(2,0, k)$ or $(2,1, k)$-asymmetric for some $k \in\left[\operatorname{rk}\left(\operatorname{core}_{t}(\lambda)\right)\right]$ and $i_{0} \neq \frac{t-1}{2}$, then

$$
\begin{align*}
& \operatorname{so}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X, y\right) \\
&=\operatorname{sgn}\left(\sigma_{\lambda}\right)(-1)^{\epsilon_{1}(\lambda)} \frac{\left(y^{-\mu_{i_{0}}^{(1)}+1} s_{\pi_{i_{0}}^{(1)}}\left(X^{t}, \bar{X}^{t}, y^{t}\right)-y^{\mu_{i_{0}}^{(1)}} s_{\pi_{i_{0}}^{(1)}}\left(X^{t}, \bar{X}^{t}, \bar{y}^{t}\right)\right)}{(y-1)} \\
& \times \prod_{\substack{i=0 \\
i \neq i_{0}}}^{\left\lfloor\frac{t-2}{2}\right\rfloor} s_{\pi_{i}^{(1)}}\left(X^{t}, \bar{X}^{t}\right) \times \begin{cases}\left.\operatorname{so}_{\lambda}{ }_{\lambda} \frac{(t-1}{2}\right)\left(X^{t}\right) & t \text { odd }, \\
1 & t \text { even },\end{cases} \tag{5.1.7}
\end{align*}
$$

where $\mu_{i_{0}}^{(1)}=t\left(\lambda_{1}^{\left(t-1-i_{0}\right)}+n_{\left(t-1-i_{0}\right)}(\lambda)-n\right)-i_{0}$.
3. If neither of the above conditions hold, then

$$
\begin{equation*}
\operatorname{so}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X, y\right)=0 \tag{5.1.8}
\end{equation*}
$$

Remark 5.5. We note that the first factor on the right side of (5.1.7) is a Laurent polynomial and approaches to $\left(1-2 \mu_{i_{0}}^{(1)}\right) s_{\pi_{i_{0}}^{(1)}}\left(X^{t}, \bar{X}^{t}, 1\right)$ as $y \rightarrow 1$.

Example 5.6. For $t=2$, Theorem 5.4 says that the character of the group $\mathrm{SO}_{7}$ of the representation indexed by the partition $(a, b, c), a \geqslant b \geqslant c \geqslant 0$, evaluated at $(x,-x, y)$ is non-zero. If $\operatorname{core}_{2}(a, b, c)$ is empty, which is $(2,0,0)$-asymmetric and $i_{0}=0$, then

$$
\begin{aligned}
& \operatorname{so}_{(a, b, c)}(x,-x, y) \\
& = \begin{cases}-\frac{y^{2-c}}{y-1} s_{\left(\frac{a+c-1}{2}, \frac{b+c}{2}\right)}\left(x^{2}, \bar{x}^{2}, y^{2}\right)+\frac{y^{c-1}}{y-1} s_{\left(\frac{a+c-1}{2}, \frac{b+c}{2}\right)}\left(x^{2}, \bar{x}^{2}, \bar{y}^{2}\right) & \text { a even, } b \text { and } c \text { odd, } \\
\frac{y^{1-b}}{y-1} s^{\left(\frac{a+b}{2}, \frac{b+c}{2}\right)}\left(x^{2}, \bar{x}^{2}, y^{2}\right)-\frac{y^{b}}{y-1} s_{\left(\frac{a+b}{2}, \frac{b+c}{2}\right)}\left(x^{2}, \bar{x}^{2}, \bar{y}^{2}\right) & a, b, c \text { even, } \\
-\frac{y^{-a}}{y-1} s_{\left(\frac{a+b}{2}, \frac{a+c+1}{2}\right)}\left(x^{2}, \bar{x}^{2}, y^{2}\right)+\frac{y^{a+1}}{y-1} s_{\left(\frac{a+b}{2}, \frac{a+c+1}{2}\right)}\left(x^{2}, \bar{x}^{2}, \bar{y}^{2}\right) & \text { a and } b \text { odd, c even. }\end{cases}
\end{aligned}
$$

If $\operatorname{core}_{2}(a, b, c)=(1)=(0 \mid 0)$, which is $(2,0,1)$-asymmetric and $i_{0}=0$, then

$$
\begin{aligned}
& \operatorname{SO}_{(a, b, c)}(x,-x, y) \\
& = \begin{cases}-\frac{y^{-a}}{y-1} s_{\left(\frac{a+c-1}{2}, \frac{a-b-1}{2}\right)}\left(x^{2}, \bar{x}^{2}, y^{2}\right)+\frac{y^{a+1}}{y-1} s_{\left(\frac{a+c-1}{2}, \frac{a-b-1}{2}\right)}\left(x^{2}, \bar{x}^{2}, \bar{y}^{2}\right) & \text { a odd, b and } c \text { even, } \\
\frac{y^{-a}}{y-1} s_{\left(\frac{a+b}{2}, \frac{a-c}{2}\right)}\left(x^{2}, \bar{x}^{2}, y^{2}\right)-\frac{y^{a+1}}{y-1} s_{\left(\frac{a+b}{2}, \frac{a-c}{2}\right)}\left(x^{2}, \bar{x}^{2}, \bar{y}^{2}\right) & \text { a, b,c odd, } \\
-\frac{y^{1-b}}{y-1} s_{\left(\frac{a+b}{2}, \frac{b-c-1}{2}\right)}\left(x^{2}, \bar{x}^{2}, y^{2}\right)+\frac{y^{b}}{y-1} s_{\left(\frac{a+b}{2}, \frac{b-c-1}{2}\right)}\left(x^{2}, \bar{x}^{2}, \bar{y}^{2}\right) & \text { a and } b \text { even, c odd. }\end{cases}
\end{aligned}
$$

If $\operatorname{core}_{2}(a, b, c)=(2,1)=(1 \mid 1)$, which is $(2,1,1)$-asymmetric and $i_{0}=0$, then

$$
\operatorname{so}_{(a, b, c)}(x,-x, y)=\frac{y^{3}}{y-1} S_{\left(\frac{a-2}{2}, \frac{b-1}{2}, \frac{c}{2}\right)}\left(x^{2}, \bar{x}^{2}, y^{2}\right)-\frac{y^{-2}}{y-1} S_{\left(\frac{a-2}{2}, \frac{b-1}{2}, \frac{c}{2}\right)}\left(x^{2}, \bar{x}^{2}, \bar{y}^{2}\right) .
$$

Lastly, if $\operatorname{core}_{2}(a, b, c)=(3,2,1)=(2,0 \mid 2,0)$ ( $a$ and $c$ are odd, and $b$ is even), which is $(2,0,1)$-asymmetric and $i_{0}=0$, then

$$
\mathrm{SO}_{(a, b, c)}(x,-x, y)=\frac{y^{-a}}{y-1} s_{\left(\frac{a-c-2}{2}, \frac{a-b-1}{2}\right)}\left(x^{2}, \bar{x}^{2}, y^{2}\right)-\frac{y^{a+1}}{y-1} s_{\left(\frac{a-c-2}{2}, \frac{a-b-1}{2}\right)}\left(x^{2}, \bar{x}^{2}, \bar{y}^{2}\right) .
$$

For the symplectic case, we take $G=\operatorname{Sp}_{2 t n+2}$, the symplectic group of $(2 t n+2) \times$ $(2 t n+2)$ matrices. If $\lambda$ is a partition such that $\operatorname{core}_{t}(\lambda)$ is either $(3,0, k)$ or $(3,2, k)$ asymmetric for some $k \in\left[\operatorname{rk}\left(\operatorname{core}_{t}(\lambda)\right)\right]$, then by Corollary 5.21, there exists a unique $i_{0} \in\left[0,\left\lfloor\frac{t-2}{2}\right\rfloor\right] \cup\{t-1\}$ such that

$$
\begin{aligned}
n_{i}(\lambda)+n_{t-2-i}(\lambda) & =\left\{\begin{array}{ll}
2 n+1+\delta_{i_{0}, \frac{t-2}{2}} & \text { if } i=i_{0}, \\
2 n & \text { otherwise, }
\end{array} \quad \text { for } \quad 0 \leqslant i \leqslant\left\lfloor\frac{t-2}{2}\right\rfloor,\right. \\
\text { and } \quad n_{t-1}(\lambda) & = \begin{cases}n+1 & \text { if } i=i_{0}, \\
n & \text { otherwise } .\end{cases}
\end{aligned}
$$

For such a partition $\lambda$, let

$$
\begin{equation*}
\epsilon_{2}(\lambda):=\sum_{i=t-i_{0}}^{t-1} n_{i-1}(\lambda)+\sum_{i=\left\lfloor\frac{t+2}{2}\right\rfloor}^{t-1}\left(\binom{n_{i-1}(\lambda)+1}{2}+(t-1) n\left(n_{i-1}(\lambda)-n\right)\right), \tag{5.1.9}
\end{equation*}
$$

and $\pi_{i}^{(2)}:=\lambda_{1}^{(t-2-i)}+\left(\lambda^{(i)}, 0, \ldots, 0,-\operatorname{rev}\left(\lambda^{(t-2-i)}\right)\right)$ has $2 n+\delta_{i, i_{0}}$ parts for $0 \leqslant i \leqslant\left\lfloor\frac{t-3}{2}\right\rfloor$. We note that the empty partition is vacuously ( $3,0,0$ )-asymmetric with $i_{0}=0$.

Theorem 5.7. Let $\lambda$ be a partition of length at most $t n+1$. The symplectic character $\operatorname{sp}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X, y\right)$ is as follows:

1. If $\operatorname{core}_{t}(\lambda)$ is either $(3,0, k)$ or $(3,2, k)$-asymmetric, for some $k \in\left[\operatorname{rk}\left(\operatorname{core}_{t}(\lambda)\right)\right]$ and $i_{0}=t-1$, then

$$
\begin{align*}
\operatorname{sp}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X, y\right)=(-1)^{\epsilon_{2}(\lambda)} & \operatorname{sgn}\left(\sigma_{\lambda}\right) \operatorname{sp}_{(t-1)}(y) \operatorname{sp}_{\lambda^{(t-1)}}\left(X^{t}, y^{t}\right) \\
& \times \prod_{i=0}^{\left\lfloor\frac{t-3}{2}\right\rfloor} s_{\pi_{i}^{(2)}}\left(X^{t}, \bar{X}^{t}\right) \times \begin{cases}\mathrm{so}_{\lambda}\left(\frac{t-2}{2}\right) \\
1 & t \text { even }, \\
1 & t \text { odd } .\end{cases} \tag{5.1.10}
\end{align*}
$$

2. If $\operatorname{core}_{t}(\lambda)$ is either $(3,0, k)$ or $(3,2, k)$-asymmetric, for some $k \in\left[\operatorname{rk}\left(\operatorname{core}_{t}(\lambda)\right)\right]$ and $i_{0}=\frac{t-2}{2}$, then

$$
\begin{align*}
& \operatorname{sp}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X, y\right)=(-1)^{\epsilon_{2}(\lambda)+n} \operatorname{sgn}\left(\sigma_{\lambda}\right) \operatorname{sp}_{\left(\frac{t-2}{2}\right)}(y) \operatorname{sp}_{\lambda^{(t-1)}}\left(X^{t}\right) \\
&\left.\times \prod_{i=0}^{\left\lfloor\frac{t-3}{2}\right\rfloor} s_{\pi_{i}^{(2)}}\left(X^{t}, \bar{X}^{t}\right) \times \operatorname{so}_{\lambda}{ }_{\lambda} \frac{t-2}{2}\right)  \tag{5.1.11}\\
&\left(X^{t}, y^{t}\right) .
\end{align*}
$$

3. If $\operatorname{core}_{t}(\lambda)$ is either $(3,0, k)$ or $(3,2, k)$-asymmetric, for some $k \in\left[\operatorname{rk}\left(\operatorname{core}_{t}(\lambda)\right)\right]$ and $i_{0} \neq t-1, \frac{t-2}{2}$, then

$$
\begin{align*}
& \operatorname{sp}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X, y\right) \\
& =\operatorname{sgn}\left(\sigma_{\lambda}\right)(-1)^{\epsilon_{2}(\lambda)} \frac{\left(y^{-\mu_{i_{0}}^{(2)}} s_{\pi_{i_{0}}^{(2)}}\left(X^{t}, \bar{X}^{t}, y^{t}\right)-y^{\mu_{i_{0}}^{(2)}} s_{\pi_{i_{0}(2)}}\left(X^{t}, \bar{X}^{t}, \bar{y}^{t}\right)\right)}{(y-\bar{y})} \\
& \quad \times \prod_{\substack{i=0 \\
i \neq i_{0}}}^{\left.\frac{t-3}{2}\right\rfloor} s_{\pi_{i}^{(2)}}\left(X^{t}, \bar{X}^{t}\right) \times \operatorname{sp}_{\lambda^{(t-1)}}\left(X^{t}\right) \times \begin{cases}\operatorname{so}_{\lambda}\left(\frac{t-2}{2}\right) \\
1 & t \text { is even, }, \\
1 & t \text { is odd },\end{cases} \tag{5.1.12}
\end{align*}
$$

where $\mu_{i_{0}}^{(2)}=t\left(\lambda_{1}^{\left(t-2-i_{0}\right)}+n_{\left(t-2-i_{0}\right)}(\lambda)-n\right)-i_{0}$.
4. If none of the above conditions hold, then

$$
\begin{equation*}
\operatorname{sp}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X, y\right)=0 \tag{5.1.13}
\end{equation*}
$$

Remark 5.8. We note that the first factor on the right side of 5.1 .12 is a Laurent polynomial and approaches to $-\mu_{i_{0}}^{(2)} s_{\pi_{i_{0}}^{(2)}}\left(X^{t}, \bar{X}^{t}, 1\right)$ as $y \rightarrow 1$.

Example 5.9. For $t=2$, Theorem 5.7 says that the character of the group $\mathrm{Sp}_{6}$ of the representation indexed by the partition $(a, b, c), a \geqslant b \geqslant c \geqslant 0$, evaluated at $(x,-x, y)$ is non-zero if and only if $a$ and $b$ have the same parity or a and $c$ have the opposite parity same as in Example 5.2. If $\operatorname{core}_{2}(a, b, c)$ is empty, which is (3, 0, 0)-asymmetric and $i_{0}=0$, then

$$
\operatorname{sp}_{(a, b, c)}(x,-x, y)= \begin{cases}\operatorname{so}_{\left(\frac{a}{2}, \frac{b+1}{2}\right)}\left(x^{2}, y^{2}\right) \operatorname{sp}_{\left(\frac{c-1}{2}\right)}\left(x^{2}\right) & \text { a even, } b \text { and } c \text { odd } \\ \operatorname{so}_{\left(\frac{a}{2}, \frac{c}{2}\right)}\left(x^{2}, y^{2}\right) \operatorname{sp}_{\left(\frac{b}{2}\right)}\left(x^{2}\right) & \text { a, b, c even } \\ -\operatorname{so}_{\left(\frac{b-1}{2}, \frac{c}{2}\right)}\left(x^{2}, y^{2}\right) \operatorname{sp}_{\left(\frac{a+1}{2}\right)}\left(x^{2}\right) & \text { a and } b \text { odd, c even }\end{cases}
$$

and if $\operatorname{core}_{2}(a, b, c)=(1)=(0 \mid 0)$, which is $(3,0,1)$-asymmetric and $i_{0}=1$, then

$$
\operatorname{sp}_{(a, b, c)}(x,-x, y)= \begin{cases}(y+\bar{y}) \operatorname{sp}_{\left(\frac{a-1}{2}, \frac{b}{2}\right)}\left(x^{2}, y^{2}\right) \mathrm{SO}_{\left(\frac{c}{2}\right)}\left(x^{2}\right) & \text { a odd, } b \text { and } c \text { even } \\ -(y+\bar{y}) \operatorname{sp}_{\left(\frac{a-1}{2}, \frac{c-1}{2}\right)}\left(x^{2}, y^{2}\right) \mathrm{SO}_{\left(\frac{b+1}{2}\right)}\left(x^{2}\right) & \text { a, b, } c \text { odd } \\ (y+\bar{y}) \operatorname{sp}_{\left(\frac{b-2}{2}, \frac{c-1}{2}\right)}\left(x^{2}, y^{2}\right) \mathrm{SO}_{\left(\frac{a+2}{2}\right)}\left(x^{2}\right) & \text { a and b even, } c \text { odd }\end{cases}
$$

For the even orthogonal case, we take $G=\mathrm{O}_{2 t n+2}$, the orthogonal group of $(2 t n+2) \times$ $(2 t n+2)$ square matrices. If $\lambda$ is a partition such that $\operatorname{core}_{t}(\lambda)$ is $(1,0, k)$-asymmetric for some $k \in\left[\operatorname{rk}\left(\operatorname{core}_{t}(\lambda)\right)\right]$, then by Corollary 5.19, there exists a unique $i_{0} \in\left[\left\lfloor\frac{t}{2}\right\rfloor\right]$ such that

$$
n_{0}(\lambda)=n, \quad n_{i}(\lambda)+n_{t-i}(\lambda)=\left\{\begin{array}{ll}
2 n+1+\delta_{i_{0}, \frac{t}{2}} & \text { if } i=i_{0} \\
2 n & \text { otherwise }
\end{array} \quad 1 \leqslant i \leqslant\left\lfloor\frac{t}{2}\right\rfloor\right.
$$

For such a partition $\lambda$, let

$$
\begin{equation*}
\epsilon_{3}(\lambda):=\left(\sum_{i=t+1-i_{0}}^{t-1} n_{i}(\lambda)\right)+\sum_{i=\left\lfloor\frac{t+2}{2}\right\rfloor}^{t-1}\left(\binom{n_{i}(\lambda)}{2}+(t-1) n\left(n_{i}(\lambda)-n\right)\right) \tag{5.1.14}
\end{equation*}
$$

and $\pi_{i}^{(3)}=\lambda_{1}^{(t-i)}+\left(\lambda^{(i)}, 0, \ldots, 0,-\operatorname{rev}\left(\lambda^{(t-i)}\right)\right)$ has $2 n+\delta_{i, i_{0}}$ parts for $1 \leqslant i \leqslant\left\lfloor\frac{t-1}{2}\right\rfloor$. Note that for $(1,0,0)$-asymmetric $t$-cores, $i_{0}=0$.

Theorem 5.10. Let $\lambda$ be a partition of length at most $t n+1$. The even orthogonal character $\mathrm{o}_{\lambda}^{\text {even }}\left(X, \omega X, \ldots, \omega^{t-1} X, y\right)$ is as follows:

1. If $\operatorname{core}_{t}(\lambda)$ is $(1,0,0)$-asymmetric, then

$$
\begin{align*}
\mathrm{o}_{\lambda}^{\text {even }}\left(X, \omega X, \ldots, \omega^{t-1} X, y\right)=(-1)^{\epsilon_{3}(\lambda)} & \operatorname{sgn}\left(\sigma_{\lambda}\right) \mathrm{o}_{\lambda}^{\text {evon }}\left(X^{t}, y^{t}\right) \prod_{i=1}^{\left\lfloor\frac{t-1}{2}\right\rfloor} s_{\pi_{i}^{(3)}}\left(X^{t}, \bar{X}^{t}\right) \\
& \times \begin{cases}(-1)^{\sum_{i} \lambda_{i}^{(t / 2)}} \mathrm{so}_{\lambda^{(t / 2)}}\left(-X^{t}\right) & t \text { even }, \\
1 & t \text { odd } .\end{cases} \tag{5.1.15}
\end{align*}
$$

2. If $\operatorname{core}_{t}(\lambda)$ is $(1,0, k)$-asymmetric for some $k \in\left[\operatorname{rk}\left(\operatorname{core}_{t}(\lambda)\right)\right]$ and $i_{0}=\frac{t}{2}$, then

$$
\begin{align*}
\mathrm{o}_{\lambda}^{\text {even }}\left(X, \omega X, \ldots, \omega^{t-1} X, y\right) & =(-1)^{\epsilon_{3}(\lambda)} \operatorname{sgn}\left(\sigma_{\lambda}\right) \mathrm{o}_{\left(\frac{t}{2}\right)}^{\text {even }}(y) \mathrm{o}_{\lambda}^{\text {evon }}\left(X^{t}\right) \\
& \times \prod_{q=1}^{\frac{t-2}{2}} s_{\pi_{q}^{(3)}}\left(X^{t}, \bar{X}^{t}\right) \times(-1)^{\sum_{i} \lambda_{i}^{(t / 2)}} \operatorname{so}_{\lambda^{(t / 2)}}\left(-X^{t},-y^{t}\right) . \tag{5.1.16}
\end{align*}
$$

3. If $\operatorname{core}_{t}(\lambda)$ is $(1,0, k)$-asymmetric for some $k \in\left[\operatorname{rk}\left(\operatorname{core}_{t}(\lambda)\right)\right]$ and $i_{0} \neq \frac{t}{2}$, then

$$
\begin{align*}
& \mathrm{o}_{\lambda}^{\text {even }}\left(X, \omega X, \ldots, \omega^{t-1} X, y\right)=(-1)^{\epsilon_{3}(\lambda)+n} \operatorname{sgn}\left(\sigma_{\lambda}\right) \mathrm{o}_{\lambda}^{\text {even }}\left(X^{t}\right) \\
& \times\left(y^{\left.-\mu_{i_{0}}^{(3)} s_{\pi_{i_{0}}^{(3)}}\left(X^{t}, \bar{X}^{t}, y^{t}\right)+y^{\mu_{i_{0}}^{(3)}} s_{\pi_{i_{0}}^{(3)}}\left(X^{t}, \bar{X}^{t}, \bar{y}^{t}\right)\right) \prod_{\substack{j=1 \\
j \neq i_{0}}}^{\left\lfloor\frac{t-1}{2}\right\rfloor} s_{\pi_{j}^{(3)}}\left(X^{t}, \bar{X}^{t}\right)}\right. \\
& \times \begin{cases}(-1)^{\sum_{i} \lambda_{i}^{(t / 2)}} \operatorname{so}_{\lambda(t / 2)}\left(-X^{t}\right) & t \text { even }, \\
1 & t \text { odd },\end{cases} \tag{5.1.17}
\end{align*}
$$

where $\mu_{i_{0}}^{(3)}=t\left(\lambda_{1}^{\left(t-i_{0}\right)}+n_{\left(t-i_{0}\right)}(\lambda)-n\right)-i_{0}$.
4. If none of the above conditions hold, then

$$
\mathrm{o}_{\lambda}^{\text {even }}\left(X, \omega X, \ldots, \omega^{t-1} X, y\right)=0
$$

Example 5.11. For $t=2$, Theorem 5.10 says that the character of the group $\mathrm{O}_{6}$ of the representation indexed by the partition $(a, b, c), a \geqslant b \geqslant c \geqslant 0$, evaluated at ( $x,-x, y$ ) is non-zero if and only if $a$ and $b$ have the same parity or a and $c$ have the opposite parity same as in Example 5.2 and Example 5.9. If $\operatorname{core}_{2}(a, b, c)$ is empty, which is (1, 0, 0)-asymmetric, then
and if $\operatorname{core}_{2}(a, b, c)=(1)=(0 \mid 0)$, which is $(1,0,1)$-asymmetric and $i_{0}=1$, then $\mathrm{O}_{(a, b, c)}^{\text {even }}(x,-x, y)= \begin{cases}(-1)^{\frac{a+b-1}{2}}(y+\bar{y}) \mathrm{So}_{\left(\frac{a-1}{2}, \frac{b}{2}\right)}\left(-x^{2},-y^{2}\right) \mathrm{O}_{\left(\frac{c}{2}\right)}^{\text {even }}\left(x^{2}\right) & \text { a odd, } b \text { and } c \text { even, } \\ (-1)^{\frac{a+c}{2}}(y+\bar{y}) \operatorname{so}_{\left(\frac{a-1}{2}, \frac{c-1}{2}\right)}^{\left(-x^{2},-y^{2}\right) \mathrm{o}_{\left(\frac{b+1}{2}\right)}^{\text {even }}\left(x^{2}\right)} & a, b, c \text { odd, }, \\ (-1)^{\frac{b+c-3}{2}}(y+\bar{y}) \mathrm{SO}_{\left(\frac{b-2}{2}, \frac{c-1}{2}\right)}\left(-x^{2},-y^{2}\right) \mathrm{o}_{\left(\frac{a+2}{2}\right)}^{\text {even }}\left(x^{2}\right) & \text { a and } b \text { even, } c \text { odd. } .\end{cases}$

Remark 5.12. The factorization of characters of classical groups of type $B, C$ and $D$ specialized with $t n$ variables are considered in Chapter 3. We will not recover the factorization results proved in Chapter 3 by substituting $y=0$ in the above factorization results as these are Laurent polynomials in $\mathbb{C}\left[x_{1}, x_{1}^{-1}, \ldots, x_{n}, x_{n}^{-1}, y, y^{-1}\right]$. (See [64]).

It is natural to ask if there are infinitely many $\left(z_{1}, z_{2}, k\right)$-asymmetric $t$-cores. Our last result answers this question in a special case. For $z_{1}>z_{2}$, let $\mathcal{Q}_{z_{1}, z_{2}}^{(t)}:=\bigcup_{k} \mathcal{Q}_{z_{1}, z_{2}, k}^{(t)}$.

Theorem 5.13. There are infinitely many $t$-cores $\mathcal{Q}_{z+2,0}^{(t)} \cup \mathcal{Q}_{z+2, z+1}^{(t)}$ for $t \geqslant z$.
This is proved in Section 5.5.

### 5.2 Background results

### 5.2.1 Properties of beta sets

We use the shorthand notations $[m]=\{1, \ldots, m\},\left[m_{1}, m_{2}\right]=\left\{m_{1}, \ldots, m_{2}\right\}$ and $m_{+}:=$ $\max (m, 0)$. We first recall a useful property of the beta numbers. For a partition $\lambda$ of length at most $m$, we see by (3.2.1) in Chapter 3:

$$
\begin{equation*}
n_{i}(\lambda, m)=n_{i}\left(\operatorname{core}_{t}(\lambda), m\right), \quad 0 \leqslant i \leqslant t-1 . \tag{5.2.1}
\end{equation*}
$$

Lemma 5.14. If $\lambda$ is a t-core of length at most $t n+1$, then

$$
\begin{equation*}
\operatorname{rk}(\lambda)=\left(n_{0}(\lambda)-n-1\right)_{+}+\sum_{i=1}^{t-1}\left(n_{i}(\lambda)-n\right)_{+} \tag{5.2.2}
\end{equation*}
$$

Proof. If $\ell(\lambda) \leqslant t n$, then using $n_{0}(\lambda, t n+1)=n_{t-1}(\lambda, t n)+1$ and $n_{i}(\lambda, t n+1)=$ $n_{i-1}(\lambda, t n), 1 \leqslant i \leqslant t-1$ in Equation (3.2.5), we see that the result holds. Assume $\ell(\lambda)=t n+1$. Since $\lambda$ is a $t$-core, $n_{0}(\lambda)=0$. Let $1 \leqslant i_{k}<\cdots<i_{1} \leqslant t-1$ such that $n_{i_{j}}(\lambda)>n$ for $1 \leqslant j \leqslant k$. Since $\lambda$ is a $t$-core, the parts of $\beta(\lambda)$ greater than $t n$ for each $j$ are:

$$
i_{j}+t n<i_{j}+t(n+1)<\cdots<i_{j}+t\left(n_{i_{j}}(\lambda)-1\right)
$$

If $r$ is the number of parts of $\beta(\lambda)$ greater than $t n$, then

$$
r=\sum_{j=1}^{k}\left(n_{i_{j}}(\lambda)-n\right)=\sum_{i=1}^{t-1}\left(n_{i}(\lambda)-n\right)_{+}
$$

Moreover, $\beta_{r}(\lambda)$ is the smallest part of $\beta(\lambda)$ greater than $t n$ and is therefore equal to $i_{k}+t n$. This implies $\lambda_{r}=\beta_{r}(\lambda)-(t n+1-r)=t n+i_{k}-(t n+1-r)=i_{k}+r-1 \geqslant r$ and $\lambda_{r+1} \leqslant t n-(t n-r) \leqslant r$, which implies the rank of $\lambda$ is $r$.

Recall the following corollary from Chapter 3.

Lemma 5.15. Let $\lambda$ be a partition of length at most tn. Then $\operatorname{core}_{t}(\lambda)$ is (1, 0,0$)$ asymmetric if and only if

$$
n_{i}(\lambda, t n)+n_{t-2-i}(\lambda, t n)=2 n, \quad 0 \leqslant i \leqslant t-2, \quad n_{t-1}(\lambda, t n)=n+1
$$

Lemma 5.16. Let $\lambda$ be a partition of length at most tn +1 . Then $\operatorname{core}_{t}(\lambda)$ is $(1,0,0)$ asymmetric if and only if

$$
n_{0}(\lambda, t n+1)=n+1, \quad n_{i}(\lambda, t n+1)+n_{t-i}(\lambda, t n+1)=2 n, \quad 1 \leqslant i \leqslant t-1
$$

Proof. As $\ell(\lambda) \leqslant t n+1 \leqslant t(n+1)$, considering $\lambda$ with $\ell(\lambda) \leqslant t n+t$, we get by Lemma 5.15.

$$
n_{i}(\lambda, t n+t)+n_{t-2-i}(\lambda, t n+t)=2 n+2, \quad 0 \leqslant i \leqslant t-2, \quad n_{t-1}(\lambda, t n+t)=n+1
$$

Now the proof of the lemma follows by noting:

$$
n_{i}(\lambda, t n+1)= \begin{cases}n_{t-1}(\lambda, t n+t) & i=0 \\ n_{i-1}(\lambda, t n+t)-1 & 1 \leqslant i \leqslant t-1\end{cases}
$$

Recall the definitions $\mathcal{Q}_{z_{1}, z_{2}, k}$ and $\mathcal{Q}_{z_{1}, z_{2}, k}^{(t)}$ from Definition 5.3.
Lemma 5.17. Let $\lambda$ be a partition of length at most $\ell$ and rank $r$. Then the following statements are equivalent.

1. $\lambda \in \mathcal{Q}_{z_{1}, z_{2}, k}$.
2. $\beta(\lambda, \ell)$ is obtained from the sequence $\left(\alpha_{1}+\ell, \ldots, \alpha_{r}+\ell, \ell-1, \ldots, 1,0\right)$ by deleting the numbers $\ell-1-z_{2}>\ell-1-z_{1}-\alpha_{r}>\cdots>\ell-1-z_{1}-\alpha_{k+1}>\ell-1-z_{1}-\alpha_{k-1}>$ $\cdots>\ell-1-z_{1}-\alpha_{1}$.

Proof. First, note that $\lambda \in \mathcal{Q}_{z_{1}, z_{2}, k}$ if and only if $\lambda$ is of the form

$$
\begin{aligned}
\lambda=\left(\alpha_{1}+1, \ldots, \alpha_{r}+r,\right. & \underbrace{r, \ldots, r}_{z_{2}}, \underbrace{r-1, \ldots, r-1}_{\alpha_{r}+z_{1}-z_{2}-1}, \underbrace{r-2, \ldots, r-2}_{\alpha_{r-1}-\alpha_{r}-1}, \ldots, \underbrace{k, \ldots, k}_{\alpha_{k+1}-\alpha_{k+2}-1}, \\
& \underbrace{k-1, \ldots, k-1}_{\alpha_{k-1}-\alpha_{k+1}-1}, \underbrace{k-2, \ldots, k-2}_{\alpha_{k-2}-\alpha_{k-1}-1} \ldots, \underbrace{1, \ldots, 1}_{\alpha_{1}-\alpha_{2}-1}
\end{aligned} . .
$$

In that case, its beta set reads as:

$$
\begin{gathered}
\beta(\lambda, \ell)=(\alpha_{1}+\ell, \ldots, \alpha_{r}+\ell, \underbrace{\ell-1, \ldots, \ell-z_{2}}_{z_{2}}, \underbrace{\ell-z_{2}-2, \ldots, \ell-\left(\alpha_{r}+z_{1}\right)}_{\alpha_{r}+z_{1}-z_{2}-1}, \\
\ell-\widehat{\alpha_{r}-z_{1}}-1, \underbrace{\ell-\alpha_{r}-z_{1}-2, \ldots, \ell-\left(\alpha_{r-1}+z_{1}\right)}_{\alpha_{r-1}-\alpha_{r}-1}, \ell-\alpha_{r-1}-z_{1}-1, \\
\ldots, \ell-\widehat{\alpha_{k+2}-} z_{1}-1, \underbrace{\ell-\alpha_{k+2}-z_{1}-2, \ldots, \ell-\left(\alpha_{k+1}+z_{1}\right)}_{\alpha_{k+1}-\alpha_{k+2}-1}, \ell-\widehat{\alpha_{k+1}-} z_{1}-1, \\
\quad \underbrace{\ell-\alpha_{k+1}-z_{1}-2, \ldots, \ell-\alpha_{k}-z_{1}, \ldots, \ell-\left(\alpha_{k-1}+z_{1}\right)}_{\alpha_{k-1}-\alpha_{k+1}-1}, \ell-\widehat{\alpha_{k-1}-z_{1}-1}, \widehat{\alpha_{1}-\alpha_{2}-1} \\
\underbrace{\ell-\alpha_{k-1}-z_{1}-2, \ldots, \ell-\left(\alpha_{k-2}+z_{1}\right)}_{\alpha_{k-2}-\alpha_{k-1}-1}, \ell-\alpha_{k-2-}-1, \ldots, \ell-\widehat{\alpha_{2}-z_{1}}-1, \\
\quad \underbrace{\ell-\ell-\widehat{\alpha_{1}-z_{1}}-1, \underbrace{\ell-\alpha_{1}-z_{1}-2, \ldots, 0}_{\alpha_{1}-1}}_{\ell-\alpha_{2}-z_{1}-2, \ldots, \ell-\left(\alpha_{1}+z_{1}\right)} .
\end{gathered}
$$

So, Item 1 and Item 2 are equivalent.

Lemma 5.18. Let $\lambda$ be a t-core of length at most $t n+1$ and $0<z+2 \leqslant t+2$. Then for $i_{0} \in\left[0,\left\lfloor\frac{t-z-1}{2}\right\rfloor\right] \cup[t-z, t-1]$,

$$
\begin{align*}
n_{i}(\lambda)+n_{t-z-1-i}(\lambda) & =\left\{\begin{array}{ll}
2 n+1+\delta_{i_{0}, \frac{t-z-1}{2}} & \text { if } i=i_{0}, \\
2 n & \text { otherwise, }
\end{array} \quad \text { for } 0 \leqslant i \leqslant\left\lfloor\frac{t-z-1}{2}\right\rfloor,\right. \\
\text { and } n_{i}(\lambda) & =\left\{\begin{array}{ll}
n+1 & \text { if } i=i_{0}, \\
n & \text { otherwise, }
\end{array} \quad \text { for } t-z \leqslant i \leqslant t-1,\right. \tag{5.2.3}
\end{align*}
$$

if and only if $\lambda \in \mathcal{Q}_{z+2,0, k}^{(t)} \cup \mathcal{Q}_{z+2, z+1, k}^{(t)}$ for some $1 \leqslant k \leqslant \operatorname{rk}(\lambda)$.
Proof. Assume (5.2.3) holds for $\lambda$. Suppose we have $0 \leqslant i_{m}<i_{m-1}<\cdots<i_{1} \leqslant t-z-1$ such that $n_{i_{j}}(\lambda)>n$ for all $j \in[m]$. Since $\lambda$ is a $t$-core, for each $j$, the parts of $\beta(\lambda)$ greater than and equal to $t n$ are:

$$
i_{j}+t\left(n_{i_{j}}(\lambda)-1\right)>\cdots>i_{j}+t(n+1)>i_{j}+t n
$$

Note that by Lemma 5.14, the rank $r$ of $\lambda$ is same as the number of parts of $\beta(\lambda)$ greater than $t n$. Let $\gamma_{s}, 1 \leqslant s \leqslant r$ be the sequence of these parts greater than $t n$ arranged in decreasing order. Note that $\gamma_{s}=\alpha_{s}+t n+1$ for some $\alpha_{s}>0,1 \leqslant s \leqslant r$. Since $n_{t-z-1-i_{j}}(\lambda) \leqslant n$ for $j \in[m], i_{j} \neq \frac{t-z-1}{2}$, the parts of $\beta(\lambda)$ lesser than $t n$ are obtained from the sequence $(t n-1, t n-2, \ldots, 0)$ by deleting the numbers
$t\left(n_{t-z-1-i_{j}}(\lambda)+1\right)-i_{j}-z-1<t\left(n_{t-z-1-i_{j}}(\lambda)+2\right)-z-i_{j}-1<\cdots<t n-i_{j}-z-1$.
Suppose $i_{0} \in[0, t-z-1]$. Then either $n_{i_{0}}(\lambda)>n$, or $n_{t-z-1-i_{0}}(\lambda)>n$. If $n_{0}(\lambda) \geqslant n$, then $t n \in \beta(\lambda)$, and the deleted numbers are $t n-z-1,2 t n-z-1-\gamma_{s}, s \in[r]$, $\gamma_{s} \neq i_{0}+t\left(n_{i_{0}}(\lambda)-1\right)$ or $t-z-1-i_{0}+t\left(n_{t-z-1-i_{0}}(\lambda)-1\right)$. So, $\beta(\lambda, t n+1)$ is obtained from the sequence $\left(\alpha_{1}+t n+1, \ldots, \alpha_{r}+t n+1, t n, \ldots, 1,0\right)$ by deleting the numbers $t n-z-1>\operatorname{tn}-z-2-\alpha_{r}>\cdots>\operatorname{tn}-z-2-\alpha_{k+1}>\operatorname{tn}-z-2-$ $\alpha_{k-1}>\cdots>t n-z-2-\alpha_{1}$. Therefore by Lemma 5.17, $\lambda \in \mathcal{Q}_{z+2, z+1, k}$. Here $k$ is the position of $t\left(n_{i_{0}}(\lambda)-1\right)+i_{0}$ or $t-z-1-i_{0}+t\left(n_{t-z-1-i_{0}}(\lambda)-1\right)$ in $\beta(\lambda, t n+1)$, because their counterpart $2 t n-z-1-t\left(n_{i_{0}}(\lambda)-1\right)-i_{0}=t n_{t-z-1-i_{0}}(\lambda)-i_{0}-z-1$ or $2 t n-z-1-t+z+1+i_{0}-t\left(n_{t-z-1-i_{0}}(\lambda)-1\right)=t\left(n_{i_{0}}(\lambda)-1\right)+i_{0}$ weren't removed from the sequence $(t n-1, t n-2, \ldots, 0)$. If $n_{0}(\lambda)<n$, then $\beta(\lambda, t n+1)$ is obtained from the sequence $\left(\alpha_{1}+t n+1, \ldots, \alpha_{r}+t n+1, t n-1, \ldots, 1,0\right)$ by deleting the numbers $t n-z-2-\alpha_{r}>\cdots>t n-z-2-\alpha_{k+1}>t n-z-2-\alpha_{k-1}>\cdots>t n-z-2-\alpha_{1}$.

Therefore by Lemma 5.17, $\lambda \in \mathcal{Q}_{z+2,0, k}$.
Suppose $i_{0} \in[t-z, t-1]$. In this case $n_{i_{0}}(\lambda)=n+1$. If $n_{0}(\lambda) \geqslant n$, then $t n \in \beta(\lambda)$. and $\beta(\lambda, t n+1)$ is obtained from the sequence $\left(\alpha_{1}+t n+1, \ldots, \alpha_{r}+t n+1, t n, t n-1, \ldots, 1,0\right)$ by deleting the numbers $t n-z-1>t n-z-2-\alpha_{r}>\cdots>t n-z-2-\alpha_{k+1}>t n-z-2-\alpha_{k-1}>$ $\cdots>t n-z-2-\alpha_{1}$. So by Lemma 5.17, $\lambda \in \mathcal{Q}_{z+2, z+1, k}$. If $n_{0}(\lambda) \leqslant n$, then $\beta(\lambda, t n+1)$ is obtained from the sequence $\left(\alpha_{1}+t n+1, \ldots, \alpha_{r}+t n+1, t n-1, \ldots, 1,0\right)$ by deleting the numbers $t n-z-2-\alpha_{r}>\cdots>t n-z-2-\alpha_{k+1}>t n-z-2-\alpha_{k-1}>\cdots>t n-z-2-\alpha_{1}$. So, $\lambda \in \mathcal{Q}_{z+2,0, k}$.

Conversely, suppose $\lambda \in \mathcal{Q}_{z+2,0, k}$ and $\operatorname{rk}(\lambda)=r$. By Lemma 5.17, $\beta(\lambda)$ is obtained from the sequence $\left(\alpha_{1}+t n+1, \ldots, \alpha_{r}+t n+1, t n-1, \ldots, 1,0\right)$ by deleting the numbers $t n-z-2-\alpha_{r}>\cdots>t n-z-2-\alpha_{k+1}>t n-z-2-\alpha_{k-1}>\cdots>t n-z-2-\alpha_{1}$. Note that if $t n-z-2-\alpha_{i} \equiv \theta_{i}(\bmod t)$, then $\alpha_{i}+t n+1 \equiv t-z-1-\theta_{i}(\bmod t)$ for $i \in[r]$. In that case $n_{t-z-1-\theta_{i}}(\lambda)$ increases by one and $n_{\theta_{i}}(\lambda)$ decreases by one. Since $\lambda$ is a $t$-core and $t n \notin \beta(\lambda), \theta_{i}$, for all $i \in[r], i \neq k$ can not be equal to $t-z-1$. If

$$
i_{0}= \begin{cases}t-z-1-\theta_{k} & \text { if } t-z-1-\theta_{k} \in\left[0,\left\lfloor\frac{t-z-1}{2}\right\rfloor\right] \cup[t-z, t-1], \\ \theta_{k} & \text { otherwise },\end{cases}
$$

then it is suffices to show that $\theta_{i} \in[0, t-z-2]$, for each $i \in[r] \backslash\{k\}$, to prove 5.2.3). We prove this successively in reverse order starting from $\theta_{r}$ and going all the way to $\theta_{1}$. Since $\lambda$ is a $t$-core, if $t n-z-2-\alpha_{r}$ does not occur in $\beta(\lambda)$, then neither does $t n-z-2-\alpha_{r}+t$. Since $t n-z-2-\alpha_{r}$ is the largest number deleted from ( $t n-1, \ldots, 0$ ) to get $\beta(\lambda)$, $t n-z-2-\alpha_{r}+t \geqslant t n$. So, $\alpha_{r}+z+2 \in[z+2, t]$; and $\theta_{r} \in[0, t-z-2]$. There is nothing to show if $\theta_{r-1}=\theta_{r}$. So, assume $\theta_{r-1} \neq \theta_{r}$. Similarly, since $\lambda$ is a $t$-core, if $t n-z-2-\alpha_{r-1}$ does not occur in $\beta(\lambda)$, then neither does $t n-z-2-\alpha_{r-1}+t$. Since $t n-z-2-\alpha_{r-1}$ is the largest number congruent to $\theta_{r-1}$ deleted from ( $t n-1, \ldots, 0$ ) to get $\beta(\lambda), \alpha_{r-1}+z+2 \in[z+2, t]$ and $\theta_{r-1} \in[0, t-z-2]$. Proceeding in this way, $\theta_{i} \in[0, t-z-2] \forall i \in[r] \backslash\{k\}$. Also, if $\lambda \in \mathcal{Q}_{z+2, z+1, k}$ then (5.2.3) holds by similar arguments.

The following three corollaries are needed in the proof of the main results. These are easily shown by applying Lemma 5.18 for $z=-1,0,1$ respectively, and using the facts that $\ell\left(\operatorname{core}_{t}(\lambda)\right) \leqslant \ell(\lambda) \leqslant t n+1$ and (5.2.1) for $m=t n+1$.

Corollary 5.19. Let $\lambda$ be a partition of length at most $t n+1$, and $i_{0} \in\left[1,\left\lfloor\frac{t}{2}\right]\right]$. Then

$$
n_{0}(\lambda)=n, n_{i}(\lambda)+n_{t-i}(\lambda)=\left\{\begin{array}{ll}
2 n+1+\delta_{i_{0}, \frac{t}{2}} & \text { if } i=i_{0}  \tag{5.2.4}\\
2 n & \text { otherwise }
\end{array} \quad \text { for } \quad 1 \leqslant i \leqslant\left\lfloor\frac{t}{2}\right\rfloor\right.
$$

if and only if $\operatorname{core}_{t}(\lambda) \in \mathcal{Q}_{1,0, k}^{(t)}$ for some $1 \leqslant k \leqslant \operatorname{rk}\left(\operatorname{core}_{t}(\lambda)\right)$.
Corollary 5.20. Let $\lambda$ be a partition of length at most tn +1 , and $i_{0} \in\left[0,\left\lfloor\frac{t-1}{2}\right\rfloor\right]$. Then

$$
n_{i}(\lambda)+n_{t-1-i}(\lambda)=\left\{\begin{array}{ll}
2 n+1+\delta_{i_{0}, \frac{t-1}{2}} & \text { if } i=i_{0}  \tag{5.2.5}\\
2 n & \text { otherwise }
\end{array} \quad \text { for } \quad 0 \leqslant i \leqslant\left\lfloor\frac{t-1}{2}\right\rfloor\right.
$$

if and only if $\operatorname{core}_{t}(\lambda) \in \mathcal{Q}_{2,0, k}^{(t)} \cup \mathcal{Q}_{2,1, k}^{(t)}$ for some $1 \leqslant k \leqslant \operatorname{rk}\left(\operatorname{core}_{t}(\lambda)\right)$.
Corollary 5.21. Let $\lambda$ be a partition of length at most tn +1 , and $i_{0} \in\left[0,\left\lfloor\frac{t-2}{2}\right\rfloor\right] \cup\{t-1\}$. Then

$$
\begin{align*}
n_{i}(\lambda)+n_{t-2-i}(\lambda) & =\left\{\begin{array}{ll}
2 n+1+\delta_{i_{0}, \frac{t-2}{2}} & \text { if } i=i_{0} \\
2 n & \text { otherwise }
\end{array} \quad \text { for } 0 \leqslant i \leqslant\left\lfloor\frac{t-2}{2}\right\rfloor,\right. \\
\text { and } n_{t-1}(\lambda) & = \begin{cases}n+1 & \text { if } i_{0}=t-1 \\
n & \text { otherwise }\end{cases} \tag{5.2.6}
\end{align*}
$$

if and only if $\operatorname{core}_{t}(\lambda) \in \mathcal{Q}_{3,0, k}^{(t)} \cup \mathcal{Q}_{3,2, k}^{(t)}$ for some $1 \leqslant k \leqslant \operatorname{rk}\left(\operatorname{core}_{t}(\lambda)\right)$.

### 5.2.2 Determinantal identities

Let $\lambda$ be a partition with $\ell(\lambda) \leqslant t n+1$. Recall, $\beta_{j}^{(p)}(\lambda)$, for $0 \leqslant p \leqslant t-1,1 \leqslant j \leqslant n_{p}(\lambda)$ are the parts of $\beta(\lambda)$ that are congruent to $p$ modulo $t$, arranged in decreasing order. Additionly, for $q \in \mathbb{Z} \cup(\mathbb{Z}+1 / 2)$, define $n \times n_{p}(\lambda)$ matrices

$$
\begin{equation*}
A_{p, q}^{\lambda}=\left(x_{i}^{\beta_{j}^{p}(\lambda)+q}\right) \underset{1 \leqslant j \leqslant n_{p}(\lambda)}{1 \leqslant i n}, \bar{A}_{p, q}^{\lambda}=\left(\bar{x}_{i}^{\beta_{j}^{p}(\lambda)+q}\right) \underset{\substack{1 \leqslant j \leqslant n \\ 1 \leqslant j n_{p}(\lambda)}}{ }, \tag{5.2.7}
\end{equation*}
$$

and $1 \times n_{p}(\lambda)$ matrices

$$
\begin{equation*}
B_{p, q}^{\lambda}=\left(y^{\beta_{j}^{p}(\lambda)+q}\right)_{1 \leqslant j \leqslant n_{p}(\lambda)}, \bar{B}_{p, q}^{\lambda}=\left(\bar{y}^{\beta_{j}^{p}(\lambda)+q}\right)_{1 \leqslant j \leqslant n_{p}(\lambda)} . \tag{5.2.8}
\end{equation*}
$$

The corresponding matrices for the empty partition are denoted by

$$
\begin{equation*}
A_{p, q}=\left(x_{i}^{t\left(n+\delta_{p, 0}-j\right)+p+q}\right) \underset{\substack{1 \leqslant i \leqslant n}}{ }, \bar{A}_{p, q}=\left(\bar{x}_{i}^{t\left(n+\delta_{p, 0}-j\right)+p+q}\right) \underset{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant n+\delta_{p, 0}}}{ }, \tag{5.2.9}
\end{equation*}
$$

and

$$
\begin{equation*}
B_{p, q}=\left(y^{t\left(n+\delta_{p, 0}-j\right)+p+q}\right)_{1 \leqslant j \leqslant n+\delta_{p, 0}}, \quad \bar{B}_{p, q}=\left(y^{t\left(n+\delta_{p, 0}-j\right)+p+q}\right)_{1 \leqslant j \leqslant n+\delta_{p, 0}} . \tag{5.2.10}
\end{equation*}
$$

In all cases, whenever $q=0$, we will omit it. For example, we will write $A_{p}^{\lambda}$ instead of $A_{p, 0}^{\lambda}$. We write down alternate formulas for the classical characters using the relation(s):

$$
t \beta_{j}\left(\lambda^{(p)}\right)=\beta_{j}^{(p)}(\lambda)-p, \quad 1 \leqslant j \leqslant n .
$$

Recall, $X^{t}=\left(x_{1}^{t}, \ldots, x_{n}^{t}\right)$. If $n_{p}(\lambda)=n$, then using 2.3.1) (2.4.5) respectively, we have:

$$
\begin{align*}
s_{\lambda^{(p)}}\left(X^{t}\right) & =\frac{\operatorname{det} A_{p,-p}^{\lambda}}{\operatorname{det}\left(x_{i}^{t(n-j)}\right)_{1 \leqslant i, j \leqslant n}},  \tag{5.2.11}\\
\operatorname{sp}_{\lambda^{(p)}}\left(X^{t}\right) & =\frac{\operatorname{det}\left(A_{p, t-p}^{\lambda}-\bar{A}_{p, t-p}^{\lambda}\right)}{\operatorname{det}\left(x_{i}^{t(n+1-j)}-\bar{x}_{i}^{t(n+1-j)}\right)_{1 \leqslant i, j \leqslant n}},  \tag{5.2.12}\\
\operatorname{so}_{\lambda(p)}\left(X^{t}\right) & =\frac{\operatorname{det}\left(A_{p, t-p}^{\lambda}-\bar{A}_{p,-p}^{\lambda}\right)}{\operatorname{det}\left(x_{i}^{t(n+1-j)}-\bar{x}_{i}^{t(n-j)}\right)_{1 \leqslant i, j \leqslant n}},  \tag{5.2.13}\\
\mathrm{o}_{\lambda^{\operatorname{evp}(p)}}\left(X^{t}\right) & =\frac{2 \operatorname{det}\left(A_{p,-p}^{\lambda}+\bar{A}_{p,-p}^{\lambda}\right)}{\left(1+\delta_{n, 0}\right) \operatorname{det}\left(x_{i}^{t(n-j)}-\bar{x}_{i}^{t(n-j)}\right)_{1 \leqslant i, j \leqslant n}} \tag{5.2.14}
\end{align*}
$$

corresponding to the Schur polynomial, the symplectic character, the even orthogonal and the odd orthogonal character. If $n_{p}(\lambda)=n+1$, then using formulas 2.3.1-2.4.5 respectively, we have:

$$
\begin{align*}
s_{\lambda(p)}\left(X^{t}, y^{t}\right)= & \frac{\operatorname{det}\left(\frac{A_{p,-p}^{\lambda}}{B_{p,-p}^{\lambda}}\right)}{\operatorname{det}\left(\frac{A_{0}}{B_{0}}\right)},  \tag{5.2.15}\\
\mathrm{sp}_{\lambda^{(p)}}\left(X^{t}, y^{t}\right)= & \frac{\operatorname{det}\left(\frac{A_{p, t-p}^{\lambda}-\bar{A}_{p, t-p}^{\lambda}}{B_{p, t-p}^{\lambda}-\bar{B}_{p, t-p}^{\lambda}}\right)}{\operatorname{det}\left(\frac{A_{0, t}-\bar{A}_{0, t}}{B_{0, t}-\bar{B}_{0, t}}\right)}  \tag{5.2.16}\\
\operatorname{so}_{\lambda(p)}\left(X^{t}, y^{t}\right)= & \frac{\operatorname{det}\left(\frac{A_{p, t-p}^{\lambda}-\bar{A}_{p,-p}^{\lambda}}{B_{p, t-p}^{\lambda}-\bar{B}_{p,-p}^{\lambda}}\right)}{\operatorname{det}\left(\frac{A_{0, t}-\bar{A}_{0}}{B_{0, t}-\bar{B}_{0}}\right)}, \tag{5.2.17}
\end{align*}
$$

$$
\begin{equation*}
\mathrm{o}_{\lambda(p)}^{\mathrm{even}}\left(X^{t}, y^{t}\right)=\frac{2 \operatorname{det}\left(\frac{A_{p,-p}^{\lambda}+\bar{A}_{p,-p}^{\lambda}}{B_{p,-p}^{\lambda}+\bar{B}_{p,-p}^{\lambda}}\right)}{\left(1+\delta_{n+1,0}\right) \operatorname{det}\left(\frac{A_{0}+\bar{A}_{0}}{B_{0}+\bar{B}_{0}}\right)} \tag{5.2.18}
\end{equation*}
$$

corresponding to the Schur polynomial, the symplectic character, the even orthogonal and the odd orthogonal character. Recall the following lemma from Chapter 3.

Lemma 5.22. Let $\lambda$ be a partition of length at most tn with $\operatorname{quo}_{t}(\lambda)=\left(\lambda^{(0)}, \ldots, \lambda^{(t-1)}\right)$. If $p, q \in\{0,1, \ldots, t-1\}$ such that $n_{p}(\lambda)+n_{q}(\lambda)=2 n$, then we define $\rho_{p, q}=\lambda_{1}^{(p)}+$ $\left(\lambda^{(q)}, 0,-\operatorname{rev}\left(\lambda^{(p)}\right)\right)$, where we pad $0^{\prime}$ s in the middle so that $\rho_{p, q}$ is of length $2 n$. Then the Schur function $s_{\rho_{p, q}}\left(X^{t}, \bar{X}^{t}\right)$ can be written as

$$
s_{\rho_{p, q}}\left(X^{t}, \bar{X}^{t}\right)=\frac{(-1)^{\frac{n_{p}(\lambda)\left(n_{p}(\lambda)-1\right)}{2}}}{(-1)^{\frac{n(n-1)}{2}}} \frac{\operatorname{det}\left(\begin{array}{c|c}
A_{q,-q}^{\lambda} & \bar{A}_{p, t-p}^{\lambda}  \tag{5.2.19}\\
\hline \bar{A}_{q,-q}^{\lambda} & A_{p, t-p}^{\lambda}
\end{array}\right)}{\operatorname{det}\left(\begin{array}{c|c}
A_{q,-q} & \bar{A}_{p, t-p} \\
\hline \bar{A}_{q,-q} & A_{p, t-p}
\end{array}\right) .}
$$

Lemma 5.23. Let $\lambda$ be a partition of length at most $t n+1$ and $0 \leqslant p, q \leqslant t-1$. If $n_{p}(\lambda)+n_{q}(\lambda)=2 n+1$, then define $\rho_{p, q}^{*}=\lambda_{1}^{(p)}+\left(\lambda^{(q)}, 0,-\operatorname{rev}\left(\lambda^{(p)}\right)\right)$, where we pad $0^{\prime} s$ in the middle so that $\rho_{p, q}^{*}$ is of length $2 n+1$. Then the Schur function $s_{\rho_{p, q}^{*}}\left(X^{t}, \bar{X}^{t}, y^{t}\right)$ can be written as

$$
s_{\rho_{p, q}^{*}}\left(X^{t}, \bar{X}^{t}, y^{t}\right)=\frac{(-1)^{\frac{n_{p}(\lambda)\left(n_{p}(\lambda)-1\right)}{2}} y^{t\left(\lambda_{1}^{(p)}+n_{p}(\lambda)\right)}}{V\left(X^{t}, \bar{X}^{t}, y^{t}\right)} \operatorname{det}\left(\begin{array}{c|c}
A_{q,-q}^{\lambda} & \bar{A}_{p, t-p}^{\lambda}  \tag{5.2.20}\\
\hline \bar{A}_{q,-q}^{\lambda} & A_{p, t-p}^{\lambda} \\
\hline B_{q,-q}^{\lambda} & \bar{B}_{p, t-p}^{\lambda}
\end{array}\right)
$$

where $V(X, \bar{X}, y):=\prod_{1 \leqslant i<j \leqslant n}\left(x_{i}-x_{j}\right)\left(x_{i}-\bar{x}_{j}\right)\left(x_{j}-\bar{x}_{i}\right)\left(\bar{x}_{i}-\bar{x}_{j}\right) \prod_{i=1}^{n}\left(x_{i}-y\right)\left(x_{i}-\bar{x}_{i}\right)\left(\bar{x}_{i}-y\right)$.

Proof. We think of the first $n_{q}(\lambda)$ parts of $\rho_{p, q}^{*}$ as coming from $\lambda^{(q)}$, and the rest from $\lambda^{(p)}$. Using the Schur polynomial expression (2.3.1) for $s_{\rho_{p, q}^{*}}\left(X^{t}, \bar{X}^{t}, y^{t}\right)$, the numerator
in the expression is
$\left.\operatorname{det}\left(\begin{array}{l|l}\left(x_{i}^{t\left(\lambda_{1}^{(p)}+\lambda_{j}^{(q)}+2 n+1-j\right)}\right) & \left(x_{i}^{1 \leqslant i \leqslant n} 1 \lambda_{1}^{(p)}-\lambda_{2 n+2-j}^{(p)}+2 n+1-j\right)\end{array}\right)_{\substack{1 \leqslant j \leqslant n_{q}(\lambda)}} \begin{array}{l}1 \leqslant i \leqslant n \\ n_{q}(\lambda)+1 \leqslant j \leqslant 2 n+1 \\ \left.\bar{x}_{i}^{t\left(\lambda_{1}^{(p)}+\lambda_{j}^{(q)}+2 n+1-j\right)}\right)_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant n_{q}(\lambda)}} \\ \hline\left(\bar{x}_{i}^{t\left(\lambda_{1}^{(p)}-\lambda_{2 n+2-j}^{(p)}+2 n+1-j\right)}\right)_{\substack{1 \leqslant i \leqslant n \\ n_{q}(\lambda)+1 \leqslant j \leqslant 2 n+1}} \\ \hline\left(y^{t\left(\lambda_{1}^{(p)}+\lambda_{j}^{(q)}+2 n+1-j\right)}\right)_{1 \leqslant j \leqslant n_{q}(\lambda)}\end{array}\right)$.

Multiplying row $i$ in the top block and middle block of the numerator by $\bar{x}_{i}^{t\left(\lambda_{1}^{(p)}+n_{p}(\lambda)\right)}$ and $x_{i}^{t\left(\lambda_{1}^{(p)}+n_{p}(\lambda)\right)}$ respectively, for each $i=1,2, \ldots, n$, the last row by $\bar{y}^{t\left(\lambda_{1}^{(p)}+n_{p}(\lambda)\right)}$ and then reversing the last $n_{p}(\lambda)$ columns, the numerator equals

$$
\begin{gathered}
(-1)^{\frac{n_{p}(\lambda)\left(n_{p}(\lambda)-1\right)}{2}} y^{t\left(\lambda_{1}^{(p)}+n_{p}(\lambda)\right)} \operatorname{det}\left(\begin{array}{ll|l}
\left(x_{i}^{t\left(\lambda_{j}^{(q)}+n_{q}-j\right)}\right)_{\substack{1 \leqslant j \leqslant n \\
1 \leqslant j \leqslant n_{q}(\lambda)}}\left(\bar{x}_{i}^{t\left(\lambda_{j}^{(p)}+n_{p}+1-j\right)}\right)_{\substack{1 \leqslant i \leqslant n \\
1 \leqslant j \leqslant n_{p}(\lambda)}}\left(\bar{x}_{i}^{t\left(\lambda_{j}^{(q)}+n_{q}-j\right)}\right)_{\substack{1 \leqslant i \leqslant n \\
1 \leqslant j \leqslant n_{q}(\lambda)}}\left(x_{i}^{t\left(\lambda_{j}^{(p)}+n_{p}+1-j\right)}\right)_{\substack{1 \leqslant i \leqslant n \\
1 \leqslant j \leqslant n_{p}(\lambda)}} \\
\hline\left(y^{t\left(\lambda_{j}^{(q)}+n_{q}-j\right)}\right)_{1 \leqslant j \leqslant n_{q}(\lambda)} & \left(\bar{y}^{t\left(\lambda_{j}^{(p)}+n_{p}+1-j\right)}\right)_{1 \leqslant j \leqslant n_{p}(\lambda)}
\end{array}\right) \\
=(-1)^{\frac{n_{p}(\lambda)\left(n_{p}(\lambda)-1\right)}{2}} y^{t\left(\lambda_{1}^{(p)}+n_{p}(\lambda)\right)} \operatorname{det}\left(\begin{array}{c|c}
A_{q,-q}^{\lambda} & \bar{A}_{p, t-p}^{\lambda} \\
\hline \bar{A}_{q,-q}^{\lambda} & A_{p, t-p}^{\lambda} \\
\hline B_{q,-q}^{\lambda} & \bar{B}_{p, t-p}^{\lambda}
\end{array}\right) .
\end{gathered}
$$

Hence, 5.2.20 holds.

Lemma 5.24 ([68, Lemma 2]). The following identities hold true.

$$
\begin{aligned}
& \operatorname{det}\left(x_{i}^{n+2-j}-\bar{x}_{i}^{n+2-j}\right)_{1 \leqslant i, j \leqslant n+1} \\
& \quad=(-1)^{n} x_{n+1}^{-n}\left(x_{n+1}-\bar{x}_{n+1}\right) \prod_{i=1}^{n}\left(x_{i}-x_{n+1}\right)\left(\bar{x}_{i}-x_{n+1}\right) \operatorname{det}\left(x_{i}^{n+1-j}-\bar{x}_{i}^{n+1-j}\right)_{1 \leqslant i, j \leqslant n} .
\end{aligned}
$$

$$
\operatorname{det}\left(x_{i}^{n+1-j}+\bar{x}_{i}^{n+1-j}\right)_{1 \leqslant i, j \leqslant n+1}
$$

$$
=(-1)^{n} x_{n+1}^{-n} \prod_{i=1}^{n}\left(x_{i}-x_{n+1}\right)\left(\bar{x}_{i}-x_{n+1}\right) \operatorname{det}\left(x_{i}^{n-j}+\bar{x}_{i}^{n-j}\right)_{1 \leqslant i, j \leqslant n}
$$

$$
\begin{aligned}
& \operatorname{det}\left(x_{i}^{n-j+3 / 2}+\bar{x}_{i}^{n-j+3 / 2}\right)_{1 \leqslant i, j \leqslant n+1} \\
& =(-1)^{n} x_{n+1}^{-n-1 / 2}\left(x_{n+1}+1\right) \prod_{i=1}^{n}\left(x_{i}-x_{n+1}\right)\left(\bar{x}_{i}-x_{n+1}\right) \operatorname{det}\left(x_{i}^{n-j+1 / 2}+\bar{x}_{i}^{n-j+1 / 2}\right)_{1 \leqslant i, j \leqslant n} .
\end{aligned}
$$

$$
\operatorname{det}\left(x_{i}^{n-j+3 / 2}-\bar{x}_{i}^{n-j+3 / 2}\right)_{1 \leqslant i, j \leqslant n+1}
$$

$$
=(-1)^{n} x_{n+1}^{-n-1 / 2}\left(x_{n+1}-1\right) \prod_{i=1}^{n}\left(x_{i}-x_{n+1}\right)\left(\bar{x}_{i}-x_{n+1}\right) \operatorname{det}\left(x_{i}^{n-j+1 / 2}-\bar{x}_{i}^{n-j+1 / 2}\right)_{1 \leqslant i, j \leqslant n} .
$$

Lemma 5.25. For $i=1, \ldots, k$, fix $\ell_{i}, m_{i} \in \mathbb{Z}^{+}$such that $1+\ell_{1}+\cdots+\ell_{k}=m_{1}+\cdots+m_{k}=$ d. Let $S_{i}$ and $T_{i}$ be matrices of order $1 \times m_{i}$ and $\ell_{i} \times m_{i}$ respectively. Define a $(k+1) \times k$ block matrix

$$
U_{k}:=\left(\begin{array}{cccc}
S_{1} & S_{2} & \ldots & S_{k} \\
T_{1} & & & \\
& T_{2} & & 0 \\
& & \ddots & \\
0 & & & T_{k}
\end{array}\right)
$$

1. If for some $i_{0} \in[k]$,

$$
m_{j}=\left\{\begin{array}{ll}
\ell_{j}+1 & j=i_{0}, \\
\ell_{j} & \text { otherwise }
\end{array} \quad 1 \leqslant j \leqslant k\right.
$$

then

$$
\begin{equation*}
\operatorname{det}\left(U_{k}\right)=(-1)^{\sum_{i<i_{0}} \ell_{i}} \operatorname{det}\left(\frac{S_{i_{0}}}{T_{i_{0}}}\right) \prod_{\substack{i=1 \\ i \neq i_{0}}}^{k} \operatorname{det}\left(T_{i}\right) \tag{5.2.21}
\end{equation*}
$$

2. Otherwise

$$
\operatorname{det}\left(U_{k}\right)=0 .
$$

Proof. It is easy to see that the lemma holds in the case when $m_{1} \geqslant \ell_{1}+1$. If $m_{1} \leqslant \ell_{1}$, then applying the blockwise row operations $R_{1} \leftrightarrow R_{2}$, we see that

$$
\operatorname{det} U_{k}= \begin{cases}0 & m_{1}<\ell_{1}, \\ (-1)^{\ell_{1}} U_{k-1} & m_{1}=\ell_{1}\end{cases}
$$

Proceeding recursively in the case $m_{1}=\ell_{1}$, 5.2.21 holds. This completes the proof.
Recall the following lemma from Chapter 3.
Lemma 5.26. Suppose $u_{1}, \ldots, u_{k}$ are positive integers summing up to kn. Further, let $\left(\gamma_{i, j}\right)_{1 \leqslant i \leqslant k, 1 \leqslant j \leqslant k+1}$ be a matrix of parameters such that $\gamma_{i, k+1}=\gamma_{i, k}, 1 \leqslant i \leqslant k$ and $\Gamma$ be the square matrix consisting of its first $k$ columns. Let $U_{j}$ and $V_{j}$ be matrices of order $n \times u_{j}$ for $j \in[k]$. Finally, define a $k n \times k n$ matrix with $k \times k$ blocks as

$$
\Pi:=\left(\begin{array}{lc}
\left.\left(\gamma_{i, 2 j-1} U_{j}-\gamma_{i, 2 j} V_{j}\right) \underset{1 \leqslant j \leqslant\left\lfloor\frac{k+1}{2}\right\rfloor}{1 \leqslant i} \right\rvert\,\left(\gamma_{i, 2 k+2-2 j} U_{j}-\gamma_{i, 2 k+1-2 j} V_{j}\right) & \begin{array}{c}
1 \leqslant i \leqslant k \\
\left\lfloor\frac{k+3}{2}\right\rfloor \leqslant j \leqslant k
\end{array}
\end{array}\right) .
$$

1. If $u_{p}+u_{k+1-p} \neq 2 n$ for some $p \in[k]$, then $\operatorname{det} \Pi=0$.
2. Else if $u_{p}+u_{k+1-p}=2 n$ for all $p \in[k]$, then

$$
\begin{equation*}
\operatorname{det} \Pi=(-1)^{\Sigma}(\operatorname{det} \Gamma)^{n} \prod_{i=1}^{\left\lfloor\frac{k+1}{2}\right\rfloor} \operatorname{det} W_{i} \tag{5.2.22}
\end{equation*}
$$

where

$$
W_{i}= \begin{cases}\left(\begin{array}{c|c}
U_{i} & -V_{k+1-i} \\
\hline-V_{i} & U_{k+1-i}
\end{array}\right) & 1 \leqslant i \leqslant\left\lfloor\frac{k}{2}\right\rfloor \\
\left(U_{\frac{k+1}{2}}-V_{\frac{k+1}{2}}\right) & k \text { odd and } i=\frac{k+1}{2}\end{cases}
$$

and

$$
\Sigma= \begin{cases}0 & k \text { even } \\ n \sum_{i=\frac{k+3}{2}}^{k} u_{i} & k \text { odd }\end{cases}
$$

Lemma 5.27. Suppose $u_{1}, \ldots, u_{k}$ are positive integers summing up to $k n+1$. Further, let $\left(\gamma_{i, j}\right)_{1 \leqslant i \leqslant k, 1 \leqslant j \leqslant k+1}$ be a matrix of parameters such that $\gamma_{i, k+1}=\gamma_{i, k}, 1 \leqslant i \leqslant k$ and $\Gamma$ be the square matrix consisting of its first $k$ columns. Let $M_{j}$ and $N_{j}$ be matrices of
order $1 \times u_{j}$ for $j \in[k]$, and $U_{j}$ and $V_{j}$ be matrices of order $n \times u_{j}$ for $j \in[k]$. Finally, define a $(k n+1) \times(k n+1)$ matrix with $(k+1) \times k$ blocks as

$$
\Delta:=\left(\begin{array}{c|c}
\left(M_{j} \pm N_{j}\right)_{1 \leqslant j \leqslant\left\lfloor\frac{k+1}{2}\right\rfloor} & \left(M_{j} \pm N_{j}\right)_{\left\lfloor\frac{k+3}{2}\right\rfloor \leqslant j \leqslant k} \\
\hline\left(\gamma_{i, 2 j-1} U_{j}-\gamma_{i, 2 j} V_{j}\right)_{\substack{1 \leqslant \backslash \leqslant k \\
1 \leqslant j \leqslant\left\lfloor\frac{k+1}{2}\right\rfloor}} & \left(\gamma_{i, 2 k+2-2 j} U_{j}-\gamma_{i, 2 k+1-2 j} V_{j}\right)_{\substack{1 \leq i \leqslant k \\
\left\lfloor\frac{k+3}{2}\right\rfloor \leqslant j \leqslant k}}
\end{array}\right) .
$$

1. If for some $1 \leqslant i_{0} \leqslant\left\lfloor\frac{k+1}{2}\right\rfloor$,

$$
u_{j}+u_{k+1-j}= \begin{cases}2 n+1+\delta_{i_{0}, \frac{k+1}{2}} & j=i_{0} \\ 2 n & \text { otherwise }\end{cases}
$$

then

$$
\begin{equation*}
\operatorname{det} \Delta=(-1)^{\chi}(\operatorname{det} \Gamma)^{n} \operatorname{det}\left(\frac{O_{i_{0}}}{W_{i_{0}}}\right) \prod_{\substack{i=1 \\ i \neq i_{0}}}^{\left\lfloor\frac{k+1}{2}\right\rfloor} \operatorname{det} W_{i}, \tag{5.2.23}
\end{equation*}
$$

where

$$
\begin{gathered}
O_{i}=\left\{\begin{array}{ll}
\left(M_{i} \pm N_{i} \mid M_{k+1-i} \pm N_{k+1-i}\right) & \text { if } 1 \leqslant i \leqslant\left\lfloor\frac{k}{2}\right\rfloor, \\
\left(M_{\frac{k+1}{2}} \pm N_{\frac{k+1}{2}}\right) & k \text { odd and } i=\frac{k+1}{2}, \\
W_{i}= \begin{cases}\left(\begin{array}{c|c}
U_{i} & -V_{k+1-i} \\
\hline-V_{i} & U_{k+1-i}
\end{array}\right) & 1 \leqslant i \leqslant\left\lfloor\frac{k}{2}\right\rfloor, \\
\left(U_{\frac{k+1}{2}}-V_{\frac{k+1}{2}}\right. & k \text { odd and } i=\frac{k+1}{2},\end{cases}
\end{array} .\left\{\begin{array}{l} 
\\
\hline
\end{array}\right.\right.
\end{gathered}
$$

and

$$
\chi=\left(\sum_{i=k+2-i_{0}}^{k} u_{i}\right)+\sum_{i=\left\lfloor\frac{k+3}{2}\right\rfloor}^{k} k n u_{i} .
$$

2. Otherwise

$$
\operatorname{det} \Delta=0
$$

Proof. Consider the permutation $\zeta$ in $S_{k n+1}$ which rearranges the columns of $\Delta$ blockwise in the following order: $1, k, 2, k-1, \ldots$. In other words, $\zeta$ can be written in one-line
notation as

$$
\begin{aligned}
\zeta= & (\underbrace{1, \ldots, u_{1}}_{u_{1}}, \underbrace{k n-u_{k}+2, \ldots, k n+1}_{u_{k}}, \\
\underbrace{u_{1}+1, \ldots, u_{1}+u_{2}}_{u_{2}}, & \underbrace{k n-u_{k}-u_{k-1}+2, \ldots, k n+1-u_{k}}_{u_{k-1}}, \ldots) .
\end{aligned}
$$

Then, the number of inversions of $\zeta$ is

$$
\begin{equation*}
\operatorname{inv}(\zeta)=\sum_{i=\left\lfloor\frac{k+3}{2}\right\rfloor}^{k} u_{i}\left(k n+1-\left(u_{1}+\cdots+u_{k+1-i}\right)-\left(u_{i}+\cdots+u_{k}\right)\right) \tag{5.2.24}
\end{equation*}
$$

Here we note that

$$
\begin{equation*}
\operatorname{det} \Delta=\operatorname{sgn}(\zeta) \operatorname{det}\left(\frac{M_{j^{\prime \prime}} \pm N_{j^{\prime \prime}}}{\gamma_{i, j} U_{j^{\prime \prime}}-\gamma_{i, j^{\prime}} V_{j^{\prime \prime}}}\right)_{1 \leqslant i, j \leqslant k} \tag{5.2.25}
\end{equation*}
$$

where

$$
j^{\prime}=j-(-1)^{j} \quad \text { and } \quad j^{\prime \prime}= \begin{cases}\frac{j+1}{2} & j \text { odd } \\ k+1-\frac{j}{2} & j \text { even }\end{cases}
$$

Now we see that

$$
\binom{M_{j^{\prime \prime}} \pm N_{j^{\prime \prime}}}{\gamma_{i, j} U_{j^{\prime \prime}}-\gamma_{i, j^{\prime}} V_{j^{\prime \prime}}}_{1 \leqslant i, j \leqslant k}=\left(\begin{array}{c|c}
1 & 0 \\
0 & \left(\gamma_{i, j} I_{n}\right)_{1 \leqslant i, j \leqslant k}
\end{array}\right) \times\left(\begin{array}{cccc}
O_{1} & O_{2} & & O_{\left\lfloor\frac{k+1}{2}\right\rfloor} \\
W_{1} & & & 0 \\
& W_{2} & & 0 \\
& & \ddots & \\
0 & & & W_{\left\lfloor\frac{k+1}{2}\right\rfloor}
\end{array}\right)
$$

Since the matrix $\left(\gamma_{i, j} I_{n}\right)_{1 \leqslant i, j \leqslant k}$ is the tensor product $\Gamma \otimes I_{n}$,

$$
\operatorname{det}\left(\begin{array}{c|c}
1 & 0 \\
\hline 0 & \left(\gamma_{i, j} I_{n}\right)_{1 \leqslant i, j \leqslant k}
\end{array}\right)=(\operatorname{det} \Gamma)^{n} .
$$

Therefore,

$$
\operatorname{det}\binom{M_{j^{\prime \prime}} \pm N_{j^{\prime \prime}}}{\gamma_{i, j} U_{j^{\prime \prime}}-\gamma_{i, j^{\prime}} V_{j^{\prime \prime}}}_{1 \leqslant i, j \leqslant k}=(\operatorname{det} \Gamma)^{n} \operatorname{det}\left(\begin{array}{cccc}
O_{1} & O_{2} & & O_{\left\lfloor\frac{k+1}{2}\right\rfloor}  \tag{5.2.26}\\
W_{1} & & & 0 \\
& W_{2} & & 0 \\
& & \ddots & \\
0 & & & W_{\left\lfloor\frac{k+1}{2}\right\rfloor}
\end{array}\right)
$$

If

$$
u_{j}+u_{k+1-j}= \begin{cases}2 n+1+\delta_{i_{0}, \frac{k+1}{2}} & j=i_{0} \\ 2 n & \text { otherwise }\end{cases}
$$

for some $1 \leqslant i_{0} \leqslant\left\lfloor\frac{k+1}{2}\right\rfloor$, then using Lemma 5.25 in (5.2.26) and substituting in (5.2.25), we have

$$
\begin{equation*}
\operatorname{det} \Delta=(-1)^{\chi}(\operatorname{det} \Gamma)^{n} \operatorname{det}\left(\frac{O_{i_{0}}}{W_{i_{0}}}\right) \prod_{\substack{i=1 \\ i \neq i_{0}}}^{\left\lfloor\frac{k+1}{2}\right\rfloor} \operatorname{det} W_{i} \text {. } \tag{5.2.27}
\end{equation*}
$$

Otherwise, using Lemma 5.25, the determinant of the last matrix in 5.2.26) is zero. Hence, by (5.2.25),

$$
\operatorname{det} \Delta=0
$$

This completes the proof.

### 5.3 Schur factorization

In this section, we give a proof of Theorem 5.1.
Lemma 5.28. Fix $0 \leqslant m \leqslant t-1$ and $0 \leqslant \nu_{m} \leqslant \cdots \leqslant \nu_{1} \leqslant t-m$. Let $\lambda$ be a partition of length at most $t n+m$. Then

$$
\operatorname{core}_{t}(\lambda)=\left(\nu_{1}, \ldots, \nu_{m}\right) \text { if and only if } n_{i}(\lambda)= \begin{cases}n+1 & \text { if } i=\nu_{i}+m-i \text { for some } i \\ n & \text { otherwise }\end{cases}
$$

Proof. It is obvious that $\operatorname{core}_{t}(\lambda)=\left(\nu_{1}, \ldots, \nu_{m}\right)$ if and only if $\beta\left(\operatorname{core}_{t}(\lambda)\right)=\left(\nu_{1}+t n+\right.$ $\left.m-1, \ldots, \nu_{m}+t n, t n-1, \ldots, 0\right)$. This further implies that $\operatorname{core}_{t}(\lambda)=\left(\nu_{1}, \ldots, \nu_{m}\right)$ if and only if

$$
n_{i}(\lambda)= \begin{cases}n+1 & \text { if } i=\nu_{i}+m-i \text { for some } i \\ n & \text { otherwise }\end{cases}
$$

Proof of Theorem 5.1. By Definition 2.3.1, we see that the desired Schur polynomial is

We first consider the case when $n_{c}(\lambda)>n+1$ for some $0 \leqslant c \leqslant t-1$. Permuting the columns of the matrix in the numerator in (5.3.1) by $\sigma_{\lambda}^{c}$ from (5.1.1) $\left(m=1, e_{1}=c\right)$, we see that the numerator in the right hand side of (5.3.1) is

$$
\operatorname{sgn}\left(\sigma_{\lambda}^{c}\right) \operatorname{det}\left(\begin{array}{c|c}
\left(\omega^{(p-1)(c)} A_{c}^{\lambda}\right)_{1 \leqslant p \leqslant t} & \left(\omega^{(p-1)(j-1)} A_{j-1}^{\lambda}\right)_{\substack{1 \leqslant p \leqslant t \\
1 \leq j \leqslant t \\
j \neq c+1}}  \tag{5.3.2}\\
\hline\left(\omega^{(p-1)(c)} B_{c}^{\lambda}\right)_{1 \leqslant p \leqslant m} & \left(\omega^{(p-1)(j-1)} B_{j-1}^{\lambda}\right)_{\substack{1 \leqslant p \leqslant m \\
1 \leqslant j \leqslant t \\
j \neq c+1}}
\end{array}\right),
$$

where

$$
A_{s}^{\lambda}=\left(x_{i}^{\beta_{j}^{(s)}(\lambda)}\right)_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant n_{s}(\lambda)}} \text { and } B_{s}^{\lambda}=\left(y^{\beta_{j}^{(s)}(\lambda)}\right)_{1 \leqslant j \leqslant n_{s}(\lambda)} \text {. }
$$

For $p \in[t]$, multiplying the rows of the $p^{\text {th }}$ block by $\omega^{(1-p) c}$ and for $p \in[m]$ and multiplying the row of the $(t+p)^{\text {th }}$ block by $\omega^{(1-p) c}$, we get

$$
\operatorname{sgn}\left(\sigma_{\lambda}^{c}\right) \operatorname{det}\left(\begin{array}{c|r}
\left(A_{c}^{\lambda}\right)_{1 \leqslant p \leqslant t} & \left(\omega^{(p-1)(j-c-1)} A_{j-1}^{\lambda}\right)_{\substack{1 \leqslant p \leqslant t \\
1 \leq j \leq t \\
j \neq c+1}}  \tag{5.3.3}\\
\hline\left(B_{c}^{\lambda}\right)_{1 \leqslant p \leqslant m} & \left(\omega^{(p-1)(j-c-1)} B_{j-1}^{\lambda}\right)_{\substack{1 \leqslant p \leqslant m \\
1 \leq j \leq t \\
j \neq c+1}}
\end{array}\right)
$$

Applying the blockwise row operations $R_{i} \rightarrow R_{i}-R_{1}$ for $i \in[2, t], R_{i} \rightarrow R_{i}-R_{t+1}$ for $i \in[t+2, t+m]$ and then permuting rows $R_{i}, i \in[2, t+1]$ cyclically, we have

$$
\operatorname{sgn}\left(\sigma_{\lambda}^{c}\right)(-1)^{t-1} \operatorname{det}\left(\begin{array}{c|c}
A_{c}^{\lambda} & \left(A_{j-1}^{\lambda}\right)_{\substack{1 \leqslant j \leqslant t \\
j \neq c+1}}  \tag{5.3.4}\\
\hline B_{c}^{\lambda} & \left(B_{j-1}^{\lambda}\right)_{\substack{1 \leqslant j \leqslant t \\
j \neq c+1}} \\
\hline(0)_{2 \leqslant p \leqslant t} & \left(\left(\omega^{(p-1)(j-c-1)}-1\right) A_{j-1}^{\lambda}\right)_{\substack{2 \leqslant p \leqslant t \\
1 \leq j \leq t \\
j \neq c+1}} \\
\hline(0)_{2 \leqslant p \leqslant m} & \left(\left(\omega^{(p-1)(j-c-1)}-1\right) B_{j-1}^{\lambda}\right)_{\substack{2 \leqslant p \leqslant m \\
1 \leq j \leq t \\
j \neq c+1}}
\end{array}\right) .
$$

Since $n_{c}(\lambda)>n+1$, the determinant in (5.3.4) is zero. Substituting in (5.3.1), we see that the required Schur polynomial vanishes. Now consider the case when $n_{i}(\lambda) \leqslant n+1$ for all $i \in[0, t-1]$. Since $\sum_{i} n_{i}(\lambda)=t n+m$, using pigeonhole principle there exist $\left\{e_{1}, \ldots, e_{m}\right\} \subset[0, t-1]$ such that $n_{e_{i}}(\lambda)=n+1, i \in[m]$. Let $e_{i}=\nu_{i}+m-i$ for all $i \in[m]$. Permuting the columns of the determinant in the numerator of 5.3.1) by $\sigma_{\lambda}^{E}$ from (5.1.1), we see that the numerator is

$$
\operatorname{sgn}\left(\sigma_{\lambda}^{\nu+\delta_{m}}\right) \operatorname{det}\left(\begin{array}{c|cc}
\left(\omega^{(p-1)\left(e_{j}\right)} A_{e_{j}}^{\lambda}\right)_{\substack{1 \leqslant p \leqslant t \\
1 \leqslant j \leqslant m}} & \left(\omega^{(p-1)(j-1)} A_{j-1}^{\lambda}\right)  \tag{5.3.5}\\
\hline\left(\omega^{(p-1)\left(e_{j}\right)} B_{e_{j}}^{\lambda}\right)_{\substack{1 \leqslant p \leqslant m \\
j \neq e_{1}+1, \ldots, e_{m}+1}}^{\substack{1 \leqslant j \leqslant m}} \mid & \left(\omega^{(p-1)(j-1)} B_{j-1}^{\lambda}\right) \underset{\substack{1 \leqslant p, j \leqslant t \\
j \neq e_{1}+1, \ldots, e_{m}+1}}{ }
\end{array}\right)
$$

Consider the permutation $\sigma^{*}$ in $S_{t n+m}$ which rearranges the rows of the numerator blockwise as: $1, t+1,2, t+2, \ldots, m, t+m, m+1, \ldots, t$. Then it can be seen that the numerator is

where the set $\left\{e_{m+1}<\cdots<e_{t}\right\}$ is same as $\{0, \ldots, t-1\} \backslash\left\{e_{i}\right\}_{i \in[m]}$ and

We note that the last determinant in (5.3.6) is non-zero if and only if

$$
n_{j}(\lambda)= \begin{cases}n+1 & \text { if } j=e_{1}, \ldots, e_{m} \\ n & \text { otherwise }\end{cases}
$$

So, by Lemma 5.28, we see that the Schur polynomial is non-zero if and only if $\operatorname{core}_{t}(\lambda)=$ $\left(\nu_{1}, \ldots, \nu_{m}\right)$. In this case, the numerator is

$$
\begin{equation*}
\operatorname{sgn}\left(\sigma^{*}\right) \operatorname{sgn}\left(\sigma_{\lambda}^{\nu+\delta_{m}}\right) \operatorname{det}\left(\Gamma_{m}(\lambda)\right) \prod_{i=1}^{m} \operatorname{det}\left(\frac{A_{e_{i}}^{\lambda}}{B_{e_{i}}^{\lambda}}\right) \prod_{\substack{i=0 \\ i \neq e_{i} i, i \in[m]}}^{t-1} \operatorname{det} A_{i}^{\lambda} . \tag{5.3.7}
\end{equation*}
$$

Permuting the columns $C_{s n+s}, \ldots, C_{t n+m}$ of $\Gamma_{m}(\lambda)$ cyclically in succession for $s=1, \ldots, m$ and then rows in the similar way, we have
$\operatorname{det} \Gamma_{m}(\lambda)=\left(\begin{array}{c|c|c}\left(\omega^{(i-1)\left(e_{j}\right)} I_{n \times n}\right)_{\substack{1 \leqslant i \leqslant t \\ 1 \leqslant j \leqslant m}} & \left(\omega^{(i-1)(j-1)} I_{n \times n}\right) \begin{array}{c}1 \leqslant i \leqslant t \\ 1 \leqslant j \leqslant t \\ j \neq e_{1}+1, \ldots, e_{m}+1 \\ \hline\end{array} & 0 \\ \hline 0 & 0 & \left(\omega^{(i-1)\left(e_{j}\right)}\right)_{1 \leqslant i, j \leqslant m}\end{array}\right)$.
Finally, we evaluate $\operatorname{det} \Gamma_{m}(\lambda)$ at the empty partition and note that

$$
\begin{equation*}
\frac{\operatorname{det} \Gamma_{m}(\lambda)}{\operatorname{det} \Gamma_{m}(\varnothing)}=s_{\operatorname{corer}_{t}(\lambda)}\left(1, \omega, \ldots, \omega^{m-1}\right) \tag{5.3.8}
\end{equation*}
$$

Since the denominator in (5.3.1) is same as the numerator evaluated at the empty partition. Evaluating (5.3.7) for the empty partition and substituting in (5.1.3) completes the proof.

### 5.4 Factorization of other classical characters

In this section, we prove Theorem 5.4, Theorem 5.7 and Theorem 5.10. Recall, the matrices $A_{p, q}^{\lambda}, \bar{A}_{p, q}^{\lambda}$ from (5.2.7) and $B_{p, q}^{\lambda}, \bar{B}_{p, q}^{\lambda}$ from (5.2.8).

### 5.4.1 odd orthogonal

Consider the $(t n+1) \times(t n+1)$ block matrix

$$
\begin{equation*}
\Delta_{1}:=\left(\frac{\left(B_{q-1,1}^{\lambda}-\bar{B}_{q-1,0}^{\lambda}\right)_{1 \leqslant q \leqslant t}}{\left(\omega^{(p-1) q} A_{q-1,1}^{\lambda}-\bar{\omega}^{(p-1)(q-1)} \bar{A}_{q-1,0}^{\lambda}\right)_{1 \leqslant p, q \leqslant t}}\right) . \tag{5.4.1}
\end{equation*}
$$

Substituting $M_{j}=B_{j-1,1}^{\lambda}, N_{j}=\bar{B}_{j-1,0}^{\lambda}, U_{j}=A_{j-1,1}^{\lambda}, V_{j}=\bar{A}_{j-1,0}^{\lambda}$ for $1 \leqslant j \leqslant t$ and

$$
\gamma_{i, j}= \begin{cases}\omega^{\frac{(i-1)(j+1)}{2}} & j \text { odd } \\ \omega^{-\frac{(i-1)(j-2)}{2}} & j \text { even }\end{cases}
$$

in Lemma 5.27 proves the following corollary.

Corollary 5.29. 1. If

$$
n_{i}(\lambda)+n_{t-1-i}(\lambda)=\left\{\begin{array}{ll}
2 n+1+\delta_{i_{0}, \frac{t-1}{2}} & \text { if } i=i_{0},  \tag{5.4.2}\\
2 n & \text { otherwise },
\end{array} \quad 0 \leqslant i \leqslant\left\lfloor\frac{t-1}{2}\right\rfloor,\right.
$$

for some $0 \leqslant i_{0} \leqslant\left\lfloor\frac{t-1}{2}\right\rfloor$, then

$$
\begin{equation*}
\operatorname{det} \Delta_{1}=(-1)^{\chi_{1}}(\operatorname{det} \Gamma)^{n} \operatorname{det}\left(\frac{O_{i_{0}}^{(1)}}{W_{i_{0}}^{(1)}}\right) \prod_{\substack{i=0 \\ i \neq i_{0}}}^{\left\lfloor\frac{t-1}{2}\right\rfloor} \operatorname{det} W_{i}^{(1)}, \tag{5.4.3}
\end{equation*}
$$

where

$$
O_{i}^{(1)}= \begin{cases}\left(B_{i, 1}^{\lambda}-\bar{B}_{i, 0}^{\lambda} \mid B_{t-1-i, 1}^{\lambda}-\bar{B}_{t-1-i, 0}^{\lambda}\right) & \text { if } 0 \leqslant i \leqslant\left\lfloor\frac{t-2}{2}\right\rfloor \\ \left(B_{\frac{t-1}{2}, 1}^{\lambda}-\bar{B}_{\frac{t-1}{2}, 0}^{\lambda}\right) & t \text { odd and } i=\frac{t-1}{2},\end{cases}
$$

$$
W_{i}^{(1)}= \begin{cases}\left(\begin{array}{c|c}
A_{i, 1}^{\lambda} & -\bar{A}_{t-1-i, 0}^{\lambda} \\
\hline-\bar{A}_{i, 0}^{\lambda} & A_{t-1-i, 1}^{\lambda}
\end{array}\right) & 0 \leqslant i \leqslant\left\lfloor\frac{t-2}{2}\right\rfloor, \\
\left(A_{\frac{t-1}{2}, 1}^{\lambda}-\bar{A}_{\frac{t-1}{2}, 0}^{\lambda}\right) & t \text { odd and } i=\frac{t-1}{2},\end{cases}
$$

and

$$
\chi_{1}=\left(\sum_{i=t+1-i_{0}}^{t} n_{i-1}(\lambda)\right)+\sum_{i=\left\lfloor\frac{t+3}{2}\right\rfloor}^{t} t n n_{i-1}(\lambda) .
$$

## 2. Otherwise

$$
\begin{equation*}
\operatorname{det} \Delta_{1}=0 \tag{5.4.4}
\end{equation*}
$$

Proof of Theorem 5.4. By (2.4.1), we see that the numerator of desired odd orthogonal character is given by

$$
\begin{equation*}
\operatorname{det}\left(\frac{\left(\left(\left(\omega^{p-1} x_{i}\right)^{\beta_{j}(\lambda)+1}-\left(\bar{\omega}^{p-1} \bar{x}_{i}\right)^{\beta_{j}(\lambda)}\right)_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant t n+1}}\right)_{1 \leqslant p \leqslant t}}{\left(y^{\beta_{j}(\lambda)+1}-\bar{y}^{\beta_{j}(\lambda)}\right)_{1 \leqslant j \leqslant t n+1}}\right) . \tag{5.4.5}
\end{equation*}
$$

Permuting the columns of the matrix in (5.4.5) by $\sigma_{\lambda}$ from (5.1.1) $\left(m=1, d_{1}=0\right)$ and then the $t+1$ row blocks of the numerator cyclically, the numerator is
$\operatorname{sgn}\left(\sigma_{\lambda}\right)(-1)^{t} \operatorname{det}\left(\begin{array}{c|c|c|c}B_{0,1}^{\lambda}-\bar{B}_{0}^{\lambda} & B_{1,1}^{\lambda}-\bar{B}_{1}^{\lambda} & \ldots & B_{t-1,1}^{\lambda}-\bar{B}_{t-1}^{\lambda} \\ \hline A_{0,1}^{\lambda}-\bar{A}_{0}^{\lambda} & A_{1,1}^{\lambda}-\bar{A}_{1}^{\lambda} & \ldots & A_{t-1,1}^{\lambda}-\bar{A}_{t-1}^{\lambda} \\ \hline \omega A_{0,1}^{\lambda}-\bar{A}_{0}^{\lambda} & \omega^{2} A_{1,1}^{\lambda}-\omega^{t-1} \bar{A}_{1}^{\lambda} & \ldots & A_{t-1,1}^{\lambda}-\omega \bar{A}_{t-1}^{\lambda} \\ \hline & & & \\ \vdots & \vdots & \ldots & \vdots \\ \hline \omega^{t-1} A_{0,1}^{\lambda}-\bar{A}_{0}^{\lambda} & \omega^{t-2} A_{1,1}^{\lambda}-\omega \bar{A}_{1}^{\lambda} & \ldots & A_{t-1,1}^{\lambda}-\omega^{t-1} \bar{A}_{t-1}^{\lambda}\end{array}\right)$,
where $A_{p, q}^{\lambda}, \bar{A}_{p}^{\lambda}, B_{p, q}^{\lambda}$ and $\bar{B}_{p}^{\lambda}$ are defined in (5.2.7) and (5.2.8). We note that the matrix in (5.4.6) is $\Delta_{1}$ defined in (5.4.1). We use Corollary 5.29 to get the determinant. Since the denominator in 2.4.1) is same as its numerator evaluated at the empty partition and $n_{0}(\varnothing, t n+1)=n+1, n_{i}(\varnothing, t n+1)=n$ for all $i \in[1, t-1]$, evaluating the numerator in (5.4.6) and then using (5.4.3), we see that the denominator of the desired odd orthogonal character is

$$
\begin{align*}
& \operatorname{sgn}\left(\sigma_{\varnothing}\right)(-1)^{t}(-1)^{\chi_{1}^{(0)}}(\operatorname{det} \Gamma)^{n} \operatorname{det}\left(\begin{array}{c|c}
B_{0,1}-\bar{B}_{0,0} & B_{t-1,1}-\bar{B}_{t-1,0} \\
\hline A_{0,1} & -\bar{A}_{t-1,0} \\
\hline-\bar{A}_{0,0} & A_{t-1,1}
\end{array}\right) \\
& \times \prod_{q=1}^{\left\lfloor\frac{t-2}{2}\right\rfloor} \operatorname{det}\left(\begin{array}{l|l}
A_{q, 1} & -\bar{A}_{t-1-q, 0} \\
\hline-\bar{A}_{q, 0} & A_{t-1-q, 1}
\end{array}\right) \times \begin{cases}\operatorname{det}\left(A_{\frac{t-1}{2}, 1}-\bar{A}_{\frac{t-1}{2}, 0}\right) & t \text { is odd } \\
1 & t \text { is even }\end{cases} \tag{5.4.7}
\end{align*}
$$

where

$$
\chi_{1}^{(0)}=\sum_{i=\left\lfloor\frac{t+3}{2}\right\rfloor}^{t} t n^{2} .
$$

If $\operatorname{core}_{t}(\lambda) \notin \mathcal{Q}_{2,0, k}^{(t)} \cup \mathcal{Q}_{2,1, k}^{(t)}$ for all $k \in\left[\operatorname{rk}\left(\operatorname{core}_{t}(\lambda)\right)\right]$, then by Corollary 5.20 and (5.4.4), the numerator in (5.4.6) is 0 . $\mathrm{So}^{2} \operatorname{so}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X, y\right)=0$. If $\operatorname{core}_{t}(\lambda) \in \mathcal{Q}_{2,0, k}^{(t)} \cup \mathcal{Q}_{2,1, k}^{(t)}$ for some $1 \leqslant k \leqslant \operatorname{rk}\left(\operatorname{core}_{t}(\lambda)\right)$, then we use Corollary 5.20 and (5.4.3) to factorize the numerator in 5.4.6.

Case 1. If $t$ is odd and $i_{0}=\frac{t-1}{2}$, then the numerator is

$$
\operatorname{sgn}\left(\sigma_{\lambda}\right)(-1)^{t}(-1)^{\chi_{1}}(\operatorname{det} \Gamma)^{n} \operatorname{det}\binom{B_{\frac{t-1}{2}, 1}^{\lambda}-\bar{B}_{\frac{t-1}{2}, 0}^{\lambda}}{\hline A_{\frac{t-1}{2}, 1}^{\lambda}-\bar{A}_{\frac{t-1}{2}, 0}^{\lambda}} \prod_{q=0}^{\frac{t-3}{2}} \operatorname{det}\left(\begin{array}{c|c}
A_{q, 1}^{\lambda} & -\bar{A}_{t-1-q, 0}^{\lambda}  \tag{5.4.8}\\
\hline-\bar{A}_{q, 0}^{\lambda} & A_{t-1-q, 1}^{\lambda}
\end{array}\right)
$$

By Lemma 5.24, we have

$$
\begin{array}{r}
\operatorname{det}\left(\begin{array}{c|c}
B_{0,1}-\bar{B}_{0,0} & B_{t-1,1}-\bar{B}_{t-1,0} \\
\hline A_{0,1} & -\bar{A}_{t-1,0} \\
\hline-\bar{A}_{0,0} & A_{t-1,1}
\end{array}\right) \\
\quad=\bar{y}^{\operatorname{tn}(y-1) \prod_{i=1}^{n}\left(\left(x_{i}^{t}-y^{t}\right)\left(\bar{x}_{i}^{t}-y^{t}\right)\right) \operatorname{det}\left(\begin{array}{c|c}
A_{0,1} & -\bar{A}_{t-1,0} \\
\hline-\bar{A}_{0,0} & A_{t-1,1}
\end{array}\right)}
\end{array}
$$

$$
\begin{aligned}
\text { and } \operatorname{det}( & \left(\frac{B_{0, \frac{t+1}{2}}-\bar{B}_{0, \frac{t-1}{2}}}{A_{0, \frac{t+1}{2}}-\bar{A}_{0, \frac{t-1}{2}}}\right) \\
& =(-1)^{n} y^{-t n+(1-t) / 2}\left(y^{t}-1\right) \prod_{i=1}^{n}\left(\left(x_{i}^{t}-y^{t}\right)\left(\bar{x}_{i}^{t}-y^{t}\right)\right) \operatorname{det}\left(A_{\frac{t-1}{2}, 1}-\bar{A}_{\frac{t-1}{2}, 0}\right) .
\end{aligned}
$$

Substituting in (5.4.7), we see that the denominator in this case is

$$
\begin{align*}
& \operatorname{sgn}\left(\sigma_{\varnothing}\right)(-1)^{t+n}(-1)^{\chi_{1}^{(0)}}(\operatorname{det} \Gamma)^{n} \frac{y-1}{y^{(1-t) / 2}\left(y^{t}-1\right)} \prod_{q=0}^{\frac{t-3}{2}} \operatorname{det}\left(\begin{array}{c|c}
A_{q, 1} & -\bar{A}_{t-1-q, 0} \\
\hline-\bar{A}_{q, 0} & A_{t-1-q, 1}
\end{array}\right)  \tag{5.4.9}\\
& \times \operatorname{det}\binom{B_{0, \frac{t+1}{2}}-\bar{B}_{0, \frac{t-1}{2}}}{\hline A_{0, \frac{t+1}{2}}-\bar{A}_{0, \frac{t-1}{2}}} .
\end{align*}
$$

For $q \in\left[0, \frac{t-3}{2}\right]$, multiplying by $x_{i}^{-q-1}$ and $\bar{x}_{i}^{-q}$ to the $i^{\text {th }}$ row in upper and lower blocks respectively for $i \in[n]$, both in numerator and denominator, by Lemma 5.22, we have

$$
\frac{\operatorname{det}\left(\begin{array}{c|c}
A_{q, 1}^{\lambda} & -\bar{A}_{t-1-q, 0}^{\lambda}  \tag{5.4.10}\\
\hline-\bar{A}_{q, 0}^{\lambda} & A_{t-1-q, 1}^{\lambda}
\end{array}\right)}{\operatorname{det}\left(\begin{array}{c|c}
A_{q, 1} & -\bar{A}_{t-1-q, 0} \\
\hline-\bar{A}_{q, 0} & A_{t-1-q, 1}
\end{array}\right)}=\frac{(-1)^{\frac{n_{t-1-q(\lambda)(n t-1-q(\lambda)+1)}^{2}}{(-1)^{\frac{n(n+1)}{2}}}} s_{\pi_{q}^{(1)}}\left(X^{t}, \bar{X}^{t}\right) . . . .}{}
$$

Taking the ratio of (5.4.8) and (5.4.9) and using (5.1.2), (5.4.10) and (5.2.17), we see that the required odd orthogonal character is

$$
\operatorname{sgn}\left(\sigma_{\lambda}\right)(-1)^{\epsilon+n} \frac{y-1}{y^{(1-t) / 2}\left(y^{t}-1\right)} \prod_{q=0}^{\frac{t-3}{2}} s_{\pi_{q}^{(1)}}\left(X^{t}, \bar{X}^{t}\right) \times \operatorname{so}_{\lambda}\left(\frac{t-1}{2}\right)\left(X^{t}\right),
$$

where

$$
\begin{align*}
\epsilon=\frac{t(t-1)}{2} \frac{n(n+1)}{2} & +\left(\sum_{i=t+1-i_{0}}^{t} n_{i-1}(\lambda)\right)+\sum_{i=\left\lfloor\frac{t+3}{2}\right\rfloor}^{t} \operatorname{tn}\left(n_{i-1}(\lambda)-n\right)  \tag{5.4.11}\\
& +\sum_{q=0}^{\left\lfloor\frac{t-2}{2}\right\rfloor}\left(\frac{n_{t-1-q}(\lambda)\left(n_{t-1-q}(\lambda)+1\right)}{2}-\frac{n(n+1)}{2}\right) .
\end{align*}
$$

Since $\frac{(t+1)(t-1)}{2} \frac{n(n+1)}{2}$ is even for odd $t$, the parity of $\epsilon$ is same as $\epsilon_{1}(\lambda)$ defined in 5.1.5).
Case 2. If $i_{0} \neq \frac{t-1}{2}$, then (5.4.3) for the determinant in (5.4.6), we see that the numerator is

$$
\begin{align*}
& \operatorname{sgn}\left(\sigma_{\lambda}\right)(-1)^{\chi_{1}+t}(\operatorname{det} \Gamma)^{n} \operatorname{det}\left(\begin{array}{c|c}
B_{i_{0}, 1}^{\lambda}-\bar{B}_{i_{0}, 0}^{\lambda} & B_{t-1-i_{0}, 1}^{\lambda}-\bar{B}_{t-1-i_{0}, 0}^{\lambda} \\
\hline A_{i_{0}, 1}^{\lambda} & -\bar{A}_{t-1-i_{0}, 0}^{\lambda} \\
\hline-\bar{A}_{i_{0}, 0}^{\lambda} & A_{t-1-i_{0}, 1}^{\lambda}
\end{array}\right)  \tag{5.4.12}\\
& \quad \times \prod_{\substack{q=0 \\
q \neq i_{0}}}^{\left\lfloor\frac{t-2}{2}\right\rfloor} \operatorname{det}\left(\begin{array}{cc}
A_{q, 1}^{\lambda} & -\bar{A}_{t-1-q, 0}^{\lambda} \\
\hline-\bar{A}_{q, 0}^{\lambda} & A_{t-1-q, 1}^{\lambda}
\end{array}\right) \times \begin{cases}\operatorname{det}\left(A_{\frac{t-1}{2}, 1}^{\lambda}-\bar{A}_{\frac{t-1}{2}, 0}^{\lambda}\right) & t \text { is odd. } \\
1 & t \text { is even. } .\end{cases}
\end{align*}
$$

For $q \in\left[0,\left\lfloor\frac{t-3}{2}\right\rfloor\right] \backslash\left\{i_{0}\right\}$, multiplying by $x_{i}^{-q-1}$ to the $i^{\text {th }}$ row in upper blocks and by $\bar{x}_{i}^{-q}$ to the $i^{\text {th }}$ row in lower blocks for $i \in[n]$, both in numerator and denominator, and then by Lemma 5.22, we have

$$
\frac{\operatorname{det}\left(\begin{array}{c|c}
A_{q, 1}^{\lambda} & -\bar{A}_{t-1-q, 0}^{\lambda}  \tag{5.4.13}\\
\hline-\bar{A}_{q, 0}^{\lambda} & A_{t-1-q, 1}^{\lambda}
\end{array}\right)}{\operatorname{det}\left(\begin{array}{c|c}
A_{q, 1} & -\bar{A}_{t-1-q, 0} \\
\hline-\bar{A}_{q, 0} & A_{t-1-q, 1}
\end{array}\right)}=\frac{(-1)^{\frac{n_{t-1-q}(\lambda)\left(n_{t-1-q}(\lambda)+1\right)}{2}}}{(-1)^{\frac{n(n+1)}{2}}} s_{\pi_{q}^{(1)}}\left(X^{t}, \bar{X}^{t}\right)
$$

Evaluating one of the factors in (5.4.7), we have

$$
\begin{array}{r}
\operatorname{det}\left(\begin{array}{c|c}
B_{0,1}-\bar{B}_{0,0} & B_{t-1,1}-\bar{B}_{t-1,0} \\
\hline A_{0,1} & -\bar{A}_{t-1,0} \\
\hline-\bar{A}_{0,0} & A_{t-1,1}
\end{array}\right) \\
=(-1)^{\frac{n(n-1)}{2}} x_{1} \ldots x_{n}\left(y^{1-t n}-y^{-t n}\right) V\left(X^{t}, \bar{X}^{t}, y^{t}\right) \\
 \tag{5.4.14}\\
=(-1)^{\frac{n(n-1)}{2}} x_{1} \ldots x_{n}\left(y^{1+t n}-y^{t n}\right) V\left(X^{t}, \bar{X}^{t}, \bar{y}^{t}\right),
\end{array}
$$

where $V\left(X^{t}, \bar{X}^{t}, \bar{y}^{t}\right)=\prod_{1 \leqslant i<j \leqslant n}\left(x_{i}^{t}-x_{j}^{t}\right)\left(x_{i}^{t}-\bar{x}_{j}^{t}\right)\left(x_{j}^{t}-\bar{x}_{i}^{t}\right)\left(\bar{x}_{i}^{t}-\bar{x}_{j}^{t}\right) \prod_{i=1}^{n}\left(x_{i}^{t}-y^{t}\right)\left(x_{i}^{t}-\right.$ $\left.\bar{x}_{i}^{t}\right)\left(\bar{x}_{i}^{t}-y^{t}\right)$. Using Lemma 5.23 and (5.4.14, we see that

$$
\begin{align*}
& \times\left(y^{-t\left(\lambda_{1}^{\left(t-1-i_{0}\right)}+n_{\left(t-1-i_{0}\right)}(\lambda)-n\right)+i_{0}+1} s_{\pi_{i_{0}}^{(1)}}\left(X^{t}, \bar{X}^{t}, y^{t}\right)\right. \\
& \left.-y^{t\left(\lambda_{1}^{\left(t-1-i_{0}\right)}+n_{t-1-i_{0}}(\lambda)-n\right)-i_{0}} s_{\pi_{i_{0}}^{(1)}}\left(X^{t}, \bar{X}^{t}, \bar{y}^{t}\right)\right) . \tag{5.4.15}
\end{align*}
$$

Thus, using (5.4.13), (5.4.15) and

$$
\frac{\operatorname{det}\left(A_{\frac{t-1}{2}, 1}^{\lambda}-\bar{A}_{\frac{t-1}{2}, 0}^{\lambda}\right)}{\operatorname{det}\left(A_{\frac{t-1}{2}, 1}-\bar{A}_{\frac{t-1}{2}, 0}\right)}=\operatorname{so}_{\lambda}\left(\frac{t-1}{2}\right)\left(X^{t}\right)
$$

the ratio of $(\sqrt{5.4 .12})$ and $(\sqrt{5.4 .7})$ is:

$$
\begin{array}{r}
\operatorname{sgn}\left(\sigma_{\lambda}\right) \frac{(-1)^{\epsilon}}{(y-1)}\left(y^{-t\left(\lambda_{1}^{\left(t-1-i_{0}\right)}+n_{\left(t-1-i_{0}\right)}(\lambda)-n\right)+i_{0}+1} s_{\pi_{i_{0}}^{(1)}}\left(X^{t}, \bar{X}^{t}, y^{t}\right)-y^{t\left(\lambda_{1}^{\left(t-1-i_{0}\right)}+n_{t-1-i_{0}}(\lambda)-n\right)-i_{0}}\right. \\
\left.s_{\pi_{i_{0}}^{(1)}}\left(X^{t}, \bar{X}^{t}, \bar{y}^{t}\right)\right) \times \prod_{\substack{i=0 \\
i \neq i_{0}}}^{\left\lfloor\frac{t-2}{2}\right\rfloor} s_{\pi_{i}^{(1)}}\left(X^{t}, \bar{X}^{t}\right) \times \begin{cases}\mathrm{so}_{\lambda}{ }_{\lambda}\left(\frac{t-1}{2}\right)\left(X^{t}\right) & t \text { is odd, } \\
1 & t \text { is even, },\end{cases}
\end{array}
$$

where $\epsilon$ is defined in 5.4.11. Since $\frac{t(t-2)}{2} \frac{n(n+1)}{2}$ is even for even $t$ and $\frac{(t+1)(t-1)}{2} \frac{n(n+1)}{2}$ is even for odd $t$, the parity of $\epsilon$ is same as $\epsilon_{1}(\lambda)$ defined in (5.1.5), completing the proof.

### 5.4.2 Symplectic characters

If $\sum_{i=0}^{t-2} n_{i}(\lambda)=(t-1) n$, then consider the $(t-1) n \times(t-1) n$ matrix

$$
\Pi_{2}=\left(\omega^{p q} A_{q-1,1}^{\lambda}-\bar{\omega}^{p q} \bar{A}_{q-1,1}^{\lambda}\right)_{1 \leqslant p, q \leqslant t-1} .
$$

Substituting $U_{j}=A_{j-1,1}^{\lambda}, V_{j}=\bar{A}_{j-1,1}^{\lambda}$ for $1 \leqslant j \leqslant t-1$ and

$$
\gamma_{i, j}= \begin{cases}\omega^{\frac{i(j+1)}{2}} & j \text { odd } \\ \omega^{-\frac{i j}{2}} & j \text { even }\end{cases}
$$

in Lemma 5.26 proves the following corollary.

Corollary 5.30. 1. If $n_{i}(\lambda)+n_{t-2-i}(\lambda) \neq 2 n$ for some $i \in\left[0,\left\lfloor\frac{t-2}{2}\right\rfloor\right]$, then $\operatorname{det} \Pi_{2}=0$.
2. If $n_{i}(\lambda)+n_{t-2-i}(\lambda)=2 n$ for all $i \in\left\{0,1, \ldots,\left\lfloor\frac{t-2}{2}\right\rfloor\right\}$, then

$$
\begin{align*}
\operatorname{det} \Pi_{2}=(-1)^{\Sigma_{2}}(\operatorname{det} \Gamma)^{n} & \prod_{q=0}^{\left\lfloor\frac{t-3}{2}\right\rfloor} \operatorname{det}\left(\begin{array}{l|l}
A_{i, 1}^{\lambda} & -\bar{A}_{t-2-i, 1}^{\lambda} \\
\hline-\bar{A}_{i, 1}^{\lambda} & A_{t-2-i, 1}^{\lambda}
\end{array}\right)  \tag{5.4.16}\\
\times & \times \begin{cases}\operatorname{det}\left(A_{\frac{t}{2}-1,1}^{\lambda}-\bar{A}_{\frac{t}{2}-1,1}^{\lambda}\right) & t \text { even }, \\
1 & t \text { odd }\end{cases}
\end{align*}
$$

where

$$
\Sigma_{2}= \begin{cases}n \sum_{q=\frac{t+2}{2}}^{t-1} n_{q-1}(\lambda) & t \text { even } \\ 0 & t \text { odd }\end{cases}
$$

If $\sum_{i=0}^{t-2} n_{i}(\lambda)=(t-1) n+1$, then consider the $(t-1) n+1 \times(t-1) n+1$ matrix

$$
\begin{equation*}
\Delta_{2}:=\left(\frac{\left(B_{q-1,1}^{\lambda}-\bar{B}_{q-1,1}^{\lambda}\right)_{1 \leqslant q \leqslant t-1}}{\left(\omega^{p q} A_{q-1,1}^{\lambda}-\bar{\omega}^{p q} \bar{A}_{q-1,1}^{\lambda}\right)_{1 \leqslant p, q \leqslant t-1}}\right) . \tag{5.4.17}
\end{equation*}
$$

Substituting $M_{j}=B_{j-1,1}^{\lambda}, N_{j}=\bar{B}_{j-1,1}^{\lambda}, U_{j}=A_{j-1,1}^{\lambda}, V_{j}=\bar{A}_{j-1,1}^{\lambda}$ for all $1 \leqslant j \leqslant t-1$ and

$$
\gamma_{i, j}= \begin{cases}\omega^{\frac{i(j+1)}{2}} & j \text { odd } \\ \omega^{-\frac{i j}{2}} & j \text { even }\end{cases}
$$

in Lemma 5.26 proves the following corollary.

Corollary 5.31. 1. If

$$
n_{j}(\lambda)+n_{t-2-j}(\lambda)=\left\{\begin{array}{ll}
2 n+1+\delta_{i_{0}, \frac{t-2}{2}} & j=i_{0}, \\
2 n & \text { otherwise },
\end{array} \quad j \in\left[0,\left\lfloor\frac{t-2}{2}\right\rfloor\right],\right.
$$

for some $i_{0} \in\left[0,\left\lfloor\frac{t-2}{2}\right\rfloor\right]$, then

$$
\begin{equation*}
\operatorname{det} \Delta_{2}=(-1)^{\chi_{2}}(\operatorname{det} \Gamma)^{n} \operatorname{det}\left(\frac{O_{i_{0}}^{(2)}}{W_{i_{0}}^{(2)}}\right) \prod_{\substack{i=0 \\ i \neq i_{0}}}^{\left\lfloor\frac{t-2}{2}\right\rfloor} \operatorname{det} W_{i}^{(2)}, \tag{5.4.18}
\end{equation*}
$$

where

$$
\begin{gathered}
O_{i}^{(2)}=\left\{\begin{array}{lc}
\left(B_{i, 1}^{\lambda}-\bar{B}_{i, 1}^{\lambda} \mid B_{t-2-i, 1}^{\lambda}-\bar{B}_{t-2-i, 1}^{\lambda}\right) & \text { if } 0 \leqslant i \leqslant\left\lfloor\frac{t-3}{2}\right\rfloor \\
\left(B_{\frac{t-2}{2}, 1}^{\lambda}-\bar{B}_{\frac{t-2}{2}, 1}^{\lambda}\right) & t \text { even and } i=\frac{t-2}{2},
\end{array}\right. \\
W_{i}^{(2)}= \begin{cases}\left(\begin{array}{cc}
A_{i, 1}^{\lambda} & -\bar{A}_{t-2-i, 1}^{\lambda} \\
\hline-\bar{A}_{i, 1}^{\lambda} \mid A_{t-2-i, 1}^{\lambda}
\end{array}\right) & 0 \leqslant i \leqslant\left\lfloor\frac{t-3}{2}\right\rfloor, \\
\left(A_{\frac{t-2}{2}, 1}^{\lambda}-\bar{A}_{\frac{t-2}{2}, 1}^{\lambda}\right) & t \text { even and } i=\frac{t-2}{2},\end{cases}
\end{gathered}
$$

and

$$
\chi_{2}=\left(\sum_{i=t-i_{0}}^{t-1} n_{i-1}(\lambda)\right)+\sum_{i=\left\lfloor\frac{t+2}{2}\right\rfloor}^{t-1}(t-1) n n_{i-1}(\lambda) .
$$

## 2. Otherwise

$$
\operatorname{det} \Delta_{2}=0
$$

Proof of Theorem 5.7. By (2.4.3), we see that the numerator of the required symplectic character is given by

$$
\begin{equation*}
\operatorname{det}\left(\frac{\left(\left(\left(\omega^{p-1} x_{i}\right)^{\beta_{j}(\lambda)+1}-\left(\bar{\omega}^{p-1} \bar{x}_{i}\right)^{\beta_{j}(\lambda)+1}\right)_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant t n+1}}\right)_{1 \leqslant p \leqslant t}}{\left(y^{\beta_{j}(\lambda)+1}-\bar{y}^{\beta_{j}(\lambda)+1}\right)_{1 \leqslant j \leqslant t n+1}}\right) . \tag{5.4.19}
\end{equation*}
$$

Permuting the columns of the determinant in the numerator by $\sigma_{\lambda}$ from 5.1.1) ( $\mathrm{m}=$ $1, d_{1}=0$ ) and applying blockwise row operations $R_{1} \rightarrow R_{1}+\cdots+R_{t}, R_{i} \rightarrow R_{i}-\frac{1}{t} R_{1}$,
$2 \leqslant i \leqslant t$ and then permute the last $t$ rows cyclically, we see that the numerator is
$\operatorname{sgn}\left(\sigma_{\lambda}\right) t^{n}(-1)^{t-1} \operatorname{det}\left(\begin{array}{c|c|c|c}0 & \ldots & 0 & A_{t-1,1}^{\lambda}-\bar{A}_{t-1,1}^{\lambda} \\ \hline & & & \\ B_{0,1}^{\lambda}-\bar{B}_{0,1}^{\lambda} & \ldots & B_{t-2,1}^{\lambda}-\bar{B}_{t-2,1}^{\lambda} & B_{t-1,1}^{\lambda}-\bar{B}_{t-1,1}^{\lambda} \\ \hline \omega A_{0,1}^{\lambda}-\omega^{t-1} \bar{A}_{0,1}^{\lambda} & \ldots & \omega^{t-1} A_{t-2,1}^{\lambda}-\omega \bar{A}_{t-2,1}^{\lambda} & 0 \\ \hline \vdots & & \ldots & \vdots \\ \hline \omega^{t-1} A_{0,1}^{\lambda}-\omega \bar{A}_{0,1}^{\lambda} & \ldots & \omega A_{t-2,1}^{\lambda}-\omega^{t-1} \bar{A}_{t-2,1}^{\lambda} & 0\end{array}\right)$,
where $A_{p, q}^{\lambda}, \bar{A}_{p, q}^{\lambda}, B_{p, q}^{\lambda}$ and $\bar{B}_{p, q}^{\lambda}$ are defined in (5.2.7) and (5.2.8). If $\operatorname{core}_{t}(\lambda) \notin \mathcal{Q}_{3,0, k}^{(t)} \cup \mathcal{Q}_{3,2, k}^{(t)}$ for all $k \in[\operatorname{rk}(\lambda)]$, then using Corollary 5.21, Corollary 5.30 and Corollary 5.31,

$$
\operatorname{sp}_{\lambda}\left(X, \omega X, \ldots, \omega^{t-1} X, y\right)=0
$$

If $\operatorname{core}_{t}(\lambda) \in \mathcal{Q}_{3,0, k}^{(t)} \cup \mathcal{Q}_{3,2, k}^{(t)}$ for some $k \in[\operatorname{rk}(\lambda)]$, then using Corollary 5.21, we factorize the numerator using Corollary 5.30 and Corollary 5.31. Since the denominator in 2.4.3) is its numerator evaluated for the empty partition, and the empty partition is vacuously $(3,0,0)$-asymmetric with $n_{0}(\varnothing, t n+1)=n+1$ and $n_{i}(\varnothing, t n+1)=n$ for all $i \in[1, t-1]$, the factorization for the denominator of required symplectic character is

$$
\begin{align*}
(-1)^{\chi_{2}^{(0)}}(\operatorname{det} \Gamma)^{n} \operatorname{sgn}\left(\sigma_{\varnothing}\right) t^{n}(-1)^{(t-1)\left(n^{2}+1\right)+n} \operatorname{det} & \left(A_{t-1,1}-\bar{A}_{t-1,1}\right) \\
\times \operatorname{det}\left(\begin{array}{c|c|c}
B_{0,1}-\bar{B}_{0,1} & B_{t-2,1}-\bar{B}_{t-2,1} \\
\hline A_{0,1} & -\bar{A}_{t-2,1} \\
\hline-\bar{A}_{0,1} & A_{t-2,1}
\end{array}\right) & \times \prod_{i=1}^{\left\lfloor\frac{t-3}{2}\right\rfloor} \operatorname{det}\left(\begin{array}{ll}
A_{i, 1} & -\bar{A}_{t-2-i, 1} \\
\hline-\bar{A}_{i, 1} & A_{t-2-i, 1}
\end{array}\right) \\
& \times \begin{cases}\left(A_{\frac{t-2}{2}, 1}-\bar{A}_{\frac{t-2}{2}, 1}\right) & t \text { even }, \\
1 & t \text { odd },\end{cases} \tag{5.4.21}
\end{align*}
$$

where

$$
\chi_{2}^{(0)}=\sum_{i=\left\lfloor\frac{t+2}{2}\right\rfloor}^{t-1}(t-1) n^{2} .
$$

Case 1. $i_{0}=t-1$. In this case $n_{t-1}(\lambda)=n+1$ and the matrix in 5.4.20 is block anti-diagonal $2 \times 2$ matrix. Using (5.4.16), the numerator in this case is

$$
\begin{align*}
& \operatorname{sgn}\left(\sigma_{\lambda}\right) t^{n}(-1)^{(t-1)\left(n^{2}+n+1\right)}(-1)^{\Sigma_{2}}(\operatorname{det} \Gamma)^{n} \operatorname{det}\left(\frac{A_{t-1,1}^{\lambda}-\bar{A}_{t-1,1}^{\lambda}}{B_{t-1,1}^{\lambda}-\bar{B}_{t-1,1}^{\lambda}}\right) \\
& \quad \times \prod_{i=0}^{\left\lfloor\frac{t-3}{2}\right\rfloor} \operatorname{det}\left(\begin{array}{ll}
A_{i, 1}^{\lambda} & -\bar{A}_{t-2-i, 1}^{\lambda} \\
\hline-\bar{A}_{i, 1}^{\lambda} & A_{t-2-i, 1}^{\lambda}
\end{array}\right) \times \begin{cases}\operatorname{det}\left(A_{\frac{t-2}{2}, 1}^{\lambda}-\bar{A}_{\frac{t-2}{2}, 1}^{\lambda}\right) & t \text { even } \\
1 & t \text { odd }\end{cases} \tag{5.4.22}
\end{align*}
$$

By Lemma 5.24, we have

$$
\begin{aligned}
& \operatorname{det}\left(\begin{array}{c|c}
B_{0,1}-\bar{B}_{0,1} & B_{t-2,1}-\bar{B}_{t-2,1} \\
\hline A_{0,1} & -\bar{A}_{t-2,1} \\
\hline-\bar{A}_{0,1} & A_{t-2,1}
\end{array}\right) \\
&=y^{-t n}(y-\bar{y}) \prod_{i=1}^{n}\left(\left(x_{i}^{t}-y^{t}\right)\left(\bar{x}_{i}^{t}-y^{t}\right)\right) \operatorname{det}\left(\begin{array}{c|c}
A_{0,1} & -\bar{A}_{t-2,1} \\
\hline-\bar{A}_{0,1} & A_{t-2,1}
\end{array}\right)
\end{aligned}
$$

and
$\operatorname{det}\left(\frac{A_{0, t}-\bar{A}_{0, t}}{B_{0, t}-\bar{B}_{0, t}}\right)=(-1)^{n} y^{-t n-t}\left(y^{2 t}-1\right) \prod_{i=1}^{n}\left(\left(x_{i}^{t}-y^{t}\right)\left(\bar{x}_{i}^{t}-y^{t}\right)\right) \operatorname{det}\left(A_{t-1,1}-\bar{A}_{t-1,1}\right)$.
Substituting in (5.4.21), the denominator in this case is

$$
\begin{align*}
& (-1)^{\chi_{2}^{(0)}+n}(\operatorname{det} \Gamma)^{n} \operatorname{sgn}\left(\sigma_{\varnothing}\right) t^{n}(-1)^{(t-1)\left(n^{2}+1\right)+n} \frac{\left(y^{2 t}-1\right)}{y^{t}(y-\bar{y})} \operatorname{det}\binom{A_{0, t}-\bar{A}_{0}}{B_{0, t}-\bar{B}_{0}}  \tag{5.4.23}\\
& \times \prod_{i=0}^{\left\lfloor\frac{t-3}{2}\right\rfloor} \operatorname{det}\left(\begin{array}{ll|}
A_{i, 1} & -\bar{A}_{t-2-i, 1} \\
\hline-\bar{A}_{i, 1} & A_{t-2-i, 1}
\end{array}\right) \times \begin{cases}\operatorname{det}\left(A_{\frac{t-2}{2}, 1}-\bar{A}_{\frac{t-2}{2}, 1}\right) & t \text { even, } \\
1 & t \text { odd. }\end{cases}
\end{align*}
$$

So, using Lemma 5.22, (5.2.13) and (5.2.16), we see that the required symplectic char-
acter, the ratio of (5.4.22) and (5.4.23), is

$$
(-1)^{\epsilon_{2}} \operatorname{sgn}\left(\sigma_{\lambda}\right) \frac{\left(y^{2 t}-1\right)}{y^{t}(y-\bar{y})} \operatorname{sp}_{\lambda^{(t-1)}}\left(X^{t}, y^{t}\right) \prod_{q=0}^{\left\lfloor\frac{t-3}{2}\right\rfloor} s_{\pi_{q}^{(2)}}\left(X^{t}, \bar{X}^{t}\right) \times \begin{cases}\mathrm{so}_{\lambda}\left(\frac{t-2}{2}\right)\left(X^{t}\right) & t \text { even } \\ 1 & t \text { odd }\end{cases}
$$

where

$$
\begin{align*}
\epsilon_{2}=\frac{t(t-1)}{2} \frac{n(n+1)}{2} & +(t-1) n+\sum_{q=0}^{\left\lfloor\frac{t-3}{2}\right\rfloor}\left(\frac{n_{t-2-q}(\lambda)\left(n_{t-2-q}(\lambda)+1\right)}{2}-\frac{n(n+1)}{2}\right) \\
& -\sum_{i=\left\lfloor\frac{t+2}{2}\right\rfloor}^{t-1}(t-1) n^{2}+ \begin{cases}n \sum_{q=\frac{t+2}{2}}^{t-1} n_{q-1}(\lambda) & t \text { even } \\
0 & t \text { odd. }\end{cases} \tag{5.4.24}
\end{align*}
$$

Since $\frac{(t+1)(t-1)}{2} \frac{n(n+1)}{2}$ is even for odd $t$ and the parity of $\frac{\left(t^{2}-2 t+2\right)}{2} \frac{n(n+1)}{2}$ is the same as $\frac{n(n+1)}{2}$ for even $t,(-1)^{\epsilon_{2}}$ is the same as $(-1)^{\epsilon_{2}(\lambda)}$, defined in (5.1.9).

If $i_{0} \neq t-1$, then $n_{t-1}=n$ and the numerator in 5.4.20 is
$\operatorname{sgn}\left(\sigma_{\lambda}\right) t^{n}(-1)^{(t-1)\left(n^{2}+1\right)+n} \operatorname{det}\left(A_{t-1,1}^{\lambda}-\bar{A}_{t-1,1}^{\lambda}\right) \operatorname{det}\left(\frac{\left(B_{q-1,1}^{\lambda}-\bar{B}_{q-1,1}^{\lambda}\right)_{1 \leqslant q \leqslant t-1}}{\left(\omega^{p q} A_{q-1,1}^{\lambda}-\bar{\omega}^{p q} \bar{A}_{q-1,1}^{\lambda}\right)_{1 \leqslant p, q \leqslant t-1}}\right)$.
The last determinant is $\Delta_{2}$ defined in (5.4.17).

Case 2. $i_{0}=\frac{t-2}{2}$, then using (5.4.18), the numerator in this case is

$$
\begin{align*}
& (-1)^{\chi_{2}}(\operatorname{det} \Gamma)^{n} \operatorname{sgn}\left(\sigma_{\lambda}\right) t^{n}(-1)^{(t-1)\left(n^{2}+1\right)+n} \operatorname{det}\left(A_{t-1,1}^{\lambda}-\bar{A}_{t-1,1}^{\lambda}\right) \\
& \quad \times \operatorname{det}\binom{B_{\frac{t}{2}-1,1}^{\lambda}-\bar{B}_{\frac{t}{2}-1,1}^{\lambda}}{\hline A_{\frac{t}{2}-1,1}^{\lambda}-\bar{A}_{\frac{t}{2}-1,1}^{\lambda}} \times \prod_{i=0}^{\frac{t-4}{2}} \operatorname{det}\left(\begin{array}{c|c}
A_{i, 1}^{\lambda} & -\bar{A}_{t-2-i, 1}^{\lambda} \\
\hline-\bar{A}_{i, 1}^{\lambda} & A_{t-2-i, 1}^{\lambda}
\end{array}\right) . \tag{5.4.25}
\end{align*}
$$

By Lemma 5.24, we have

$$
\begin{aligned}
& \operatorname{det}\left(\begin{array}{c|c}
B_{0,1}-\bar{B}_{0,1} & B_{t-2,1}-\bar{B}_{t-2,1} \\
\hline A_{0,1} & -\bar{A}_{t-2,1} \\
\hline-\bar{A}_{0,1} & A_{t-2,1}
\end{array}\right) \\
&=y^{-t n}(y-\bar{y}) \prod_{i=1}^{n}\left(\left(x_{i}^{t}-y^{t}\right)\left(\bar{x}_{i}^{t}-y^{t}\right)\right) \operatorname{det}\left(\begin{array}{c|c}
A_{0,1} & -\bar{A}_{t-2,1} \\
\hline-\bar{A}_{0,1} & A_{t-2,1}
\end{array}\right)
\end{aligned}
$$

and
$\operatorname{det}\left(\frac{A_{0, \frac{t}{2}}-\bar{A}_{0, \frac{t}{2}}}{B_{0, \frac{t}{2}}-\bar{B}_{0, \frac{t}{2}}}\right)=(-1)^{n} y^{-t n-t / 2}\left(y^{t}-1\right) \prod_{i=1}^{n}\left(x_{i}^{t}-y^{t}\right)\left(\bar{x}_{i}^{t}-y^{t}\right) \operatorname{det}\left(A_{\frac{t-2}{2}, 1}-\bar{A}_{\frac{t-2}{2}, 1}\right)$.
Substituting in (5.4.21), the denominator in this case is

$$
\begin{array}{r}
(-1)^{\chi_{2}^{(0)}+n}(\operatorname{det} \Gamma)^{n} \operatorname{sgn}\left(\sigma_{\varnothing}\right) t^{n}(-1)^{(t-1)\left(n^{2}+1\right)+n} \frac{\left(y^{t}-1\right)}{y^{t / 2}(y-\bar{y})} \operatorname{det}\left(A_{t-1,1}^{\lambda}-\bar{A}_{t-1,1}^{\lambda}\right) \\
\quad \times \prod_{i=0}^{\left\lfloor\frac{t-3}{2}\right\rfloor} \operatorname{det}\left(\begin{array}{l|l}
A_{i, 1} & -\bar{A}_{t-2-i, 1} \\
\hline-\bar{A}_{i, 1} & A_{t-2-i, 1}
\end{array}\right) \times \begin{cases}\operatorname{det}\binom{A_{0, \frac{t}{2}}-\bar{A}_{0, \frac{t}{2}}}{B_{0, \frac{t}{2}}-\bar{B}_{0, \frac{t}{2}}} & t \text { even, } \\
1 & t \text { odd. }\end{cases} \tag{5.4.26}
\end{array}
$$

So, using Lemma 5.22, (5.2.12) and (5.2.17), we see that the required symplectic character, the ratio of (5.4.25) and (5.4.26) is

$$
(-1)^{\epsilon_{2}^{\prime}+n} \operatorname{sgn}\left(\sigma_{\lambda}\right) \frac{\left(y^{t}-1\right)}{y^{t / 2}(y-\bar{y})} \operatorname{sp}_{\lambda^{(t-1)}}\left(X^{t}\right) \prod_{q=0}^{\left\lfloor\frac{t-3}{2}\right\rfloor} s_{\pi_{q}^{(2)}}\left(X^{t}, \bar{X}^{t}\right) \times \operatorname{so}_{\lambda}^{\left(\frac{t-2}{2}\right)}\left(X^{t}, y^{t}\right),
$$

where

$$
\begin{align*}
\epsilon_{2}^{\prime}=\frac{t(t-1)}{2} \frac{n(n+1)}{2} & +\sum_{i=t-i_{0}}^{t-1} n_{i-1}(\lambda)+\sum_{i=\left\lfloor\frac{t+2}{2}\right\rfloor}^{t-1}(t-1) n\left(n_{i-1}(\lambda)-n\right) \\
& +\sum_{q=0}^{\left\lfloor\frac{t-3}{2}\right\rfloor}\left(\frac{n_{t-2-q}(\lambda)\left(n_{t-2-q}(\lambda)+1\right)}{2}-\frac{n(n+1)}{2}\right) . \tag{5.4.27}
\end{align*}
$$

Since the parity of $\frac{\left(t^{2}-2 t+2\right)}{2} \frac{n(n+1)}{2}$ is the same as $\frac{n(n+1)}{2}$ for even $t,(-1)^{\epsilon_{2}^{\prime}}$ is the same as $(-1)^{\epsilon_{2}(\lambda)+n+1}$, defined in (5.1.9).

Case 3. $i_{0} \neq \frac{t-2}{2}$. Using (5.4.18), the numerator in this case is

$$
\begin{align*}
& (-1)^{\chi_{2}(\operatorname{det} \Gamma)^{n} \operatorname{sgn}\left(\sigma_{\lambda}\right) t^{n}(-1)^{(t-1)\left(n^{2}+1\right)+n} \operatorname{det}\left(A_{t-1,1}^{\lambda}-\bar{A}_{t-1,1}^{\lambda}\right)} \\
& \quad \times \operatorname{det}\left(\begin{array}{c|c|c}
B_{i_{0}, 1}^{\lambda}-\bar{B}_{i_{0}, 1}^{\lambda} & B_{t-2-i_{0}, 1}^{\lambda}-\bar{B}_{t-2-i_{0}, 1}^{\lambda} \\
\hline A_{i_{0}, 1}^{\lambda} & -\bar{A}_{t-2-i_{0}, 1}^{\lambda} \\
\hline-\bar{A}_{i_{0}, 1}^{\lambda} & A_{t-2-i_{0}, 1}^{\lambda}
\end{array}\right) \times \prod_{\substack{i=0 \\
i \neq i_{0}}}^{\left\lfloor\frac{t-3}{2}\right\rfloor} \operatorname{det}\left(\begin{array}{ll}
A_{i, 1}^{\lambda} & -\bar{A}_{t-2-i, 1}^{\lambda} \\
\hline-\bar{A}_{i, 1}^{\lambda} & A_{t-2-i, 1}^{\lambda}
\end{array}\right) \\
&  \tag{5.4.28}\\
&
\end{aligned} \begin{aligned}
& \times \begin{cases}\left(A_{\frac{t-2}{2}, 1}^{\lambda}-\bar{A}_{\frac{t-2}{2}, 1}^{\lambda}\right) & t \text { even, } \\
1 & t \text { odd. }\end{cases}
\end{align*}
$$

By Lemma 5.24, we have

$$
\begin{array}{r}
\operatorname{det}\left(\begin{array}{c|c}
B_{0,1}-\bar{B}_{0,1} & B_{t-2,1}-\bar{B}_{t-2,1} \\
\hline A_{0,1} & -\bar{A}_{t-2,1} \\
\hline-\bar{A}_{0,1} & A_{t-2,1}
\end{array}\right) \\
=(-1)^{\frac{n(n-1)}{2}} x_{1} \ldots x_{n}\left(y^{1-t n}-y^{-t n-1}\right) V\left(X^{t}, \bar{X}^{t}, y^{t}\right) \\
=(-1)^{\frac{n(n-1)}{2}} x_{1} \ldots x_{n}\left(y^{1+t n}-y^{t n-1}\right) V\left(X^{t}, \bar{X}^{t}, \bar{y}^{t}\right), \tag{5.4.29}
\end{array}
$$

where $V\left(X^{t}, \bar{X}^{t}, \bar{y}^{t}\right)=\prod_{1 \leqslant i<j \leqslant n}\left(x_{i}^{t}-x_{j}^{t}\right)\left(x_{i}^{t}-\bar{x}_{j}^{t}\right)\left(x_{j}^{t}-\bar{x}_{i}^{t}\right)\left(\bar{x}_{i}^{t}-\bar{x}_{j}^{t}\right) \prod_{i=1}^{n}\left(x_{i}^{t}-y^{t}\right)\left(x_{i}^{t}-\right.$ $\left.\bar{x}_{i}^{t}\right)\left(\bar{x}_{i}^{t}-y^{t}\right)$. Using Lemma 5.23 and (5.4.29), we see that

$$
\begin{align*}
& \times\left(y^{-t\left(\lambda_{1}^{\left(t-2-i_{0}\right)}+n_{\left(t-2-i_{0}\right)}(\lambda)-n\right)+i_{0}} S_{\pi_{i_{0}}^{(1)}}\left(X^{t}, \bar{X}^{t}, y^{t}\right)\right. \\
& \left.-y^{t\left(\lambda_{1}^{\left(t-1-i_{0}\right)}+n_{t-1-i_{0}}(\lambda)-n\right)-i_{0}} s_{\pi_{i_{0}}^{(1)}}\left(X^{t}, \bar{X}^{t}, \bar{y}^{t}\right)\right) . \tag{5.4.30}
\end{align*}
$$

Thus, using Lemma 5.22, 5.4.30) and

$$
\frac{\operatorname{det}\left(A_{\frac{t-2}{2}, 1}^{\lambda}-\bar{A}_{\frac{t-2}{2}, 1}^{\lambda}\right)}{\operatorname{det}\left(A_{\frac{t-2}{2}, 1}-\bar{A}_{\frac{t-2}{2}, 1}\right)}=\operatorname{so}_{\lambda}\left(\frac{t-2}{2}\right)\left(X^{t}\right)
$$

the ratio of (5.4.28) and (5.4.21) is:

$$
\begin{aligned}
\operatorname{sgn}\left(\sigma_{\lambda}\right) \frac{(-1)^{\epsilon_{2}^{\prime}}}{y-\bar{y}}\left(y^{-t\left(\lambda_{1}^{\left(t-2-i_{0}\right)}+n_{\left(t-2-i_{0}\right)}(\lambda)-n\right)+i_{0}} s_{\pi_{i_{0}}^{(2)}}\left(X^{t}, \bar{X}^{t}, y^{t}\right)-y^{t\left(\lambda_{1}^{\left(t-2-i_{0}\right)}+n_{t-2-i_{0}}(\lambda)-n\right)-i_{0}}\right. \\
\left.s_{\pi_{i_{0}}^{(2)}}\left(X^{t}, \bar{X}^{t}, \bar{y}^{t}\right)\right) \times \prod_{\substack{i=0 \\
i \neq i_{0}}}^{\left\lfloor\frac{t-2}{2}\right\rfloor} s_{\pi_{i}^{(2)}}\left(X^{t}, \bar{X}^{t}\right) \times \begin{cases}\mathrm{so}_{\lambda^{(t-1}}\left(X^{t-1}\right) \\
1 & t \text { is odd, } \\
1 & t \text { is even, }\end{cases}
\end{aligned}
$$

where $\epsilon_{2}^{\prime}$ is defined in (5.4.27). Since $\frac{(t+1)(t-1)}{2} \frac{n(n+1)}{2}$ is even for odd $t,(-1)^{\epsilon_{2}^{\prime}}$ is the same as $(-1)^{\epsilon_{2}(\lambda)}$, defined in (5.1.9). This completes the proof.

### 5.4.3 Even orthogonal characters

If $\sum_{i=1}^{t-1} n_{i}(\lambda)=(t-1) n$, then consider the $(t-1) n \times(t-1) n$ block matrix

$$
\begin{equation*}
\Pi_{3}:=\left(\omega^{p q} A_{q}^{\lambda}+\bar{\omega}^{p q} \bar{A}_{q}^{\lambda}\right)_{1 \leqslant p, q \leqslant t-1} \tag{5.4.31}
\end{equation*}
$$

Substituting $U_{j}=A_{j}^{\lambda}, V_{j}=-\bar{A}_{j}^{\lambda}$ and for $1 \leqslant j \leqslant t-1$,

$$
\gamma_{i, j}= \begin{cases}\omega^{\frac{i(j+1)}{2}} & j \text { odd } \\ \omega^{-\frac{i j}{2}} & j \text { even },\end{cases}
$$

in Lemma 5.26, we get the following corollary.
Corollary 5.32. 1. If $n_{i}(\lambda)+n_{t-i}(\lambda) \neq 2 n$ for some $i \in\left[\left\lfloor\frac{t}{2}\right\rfloor\right]$, then $\operatorname{det} \Pi_{3}=0$.
2. If $n_{i}(\lambda)+n_{t-i}(\lambda)=2 n$ for all $i \in\left[\left\lfloor\frac{t}{2}\right]\right]$, then

$$
\begin{align*}
\operatorname{det} \Pi_{3}=(-1)^{\Sigma_{3}}\left(\operatorname{det}\left(\gamma_{i, j}\right)_{1 \leqslant i, j \leqslant t-1}\right)^{n} & \prod_{q=1}^{\left\lfloor\frac{t-1}{2}\right\rfloor} \operatorname{det}\left(\begin{array}{c|c}
A_{q}^{\lambda} & \bar{A}_{t-q}^{\lambda} \\
\hline \bar{A}_{q}^{\lambda} & A_{t-q}^{\lambda}
\end{array}\right)  \tag{5.4.32}\\
& \times \begin{cases}\operatorname{det}\left(A_{\frac{t}{2}}^{\lambda}+\bar{A}_{\frac{t}{2}}^{\lambda}\right) & t \text { even }, \\
1 & t \text { odd },\end{cases}
\end{align*}
$$

where

$$
\Sigma_{3}= \begin{cases}n \sum_{q=\frac{t+2}{2}}^{t-1} n_{q}(\lambda) & t \text { even } \\ 0 & t \text { odd }\end{cases}
$$

If $\sum_{i=1}^{t-1} n_{i}(\lambda)=(t-1) n+1$, then consider the $((t-1) n+1) \times((t-1) n+1)$ block matrix

$$
\begin{equation*}
\Delta_{3}:=\left(\frac{\left(B_{q}^{\lambda}+\bar{B}_{q}^{\lambda}\right)_{1 \leqslant q \leqslant t-1}}{\left(\omega^{p q} A_{q}^{\lambda}+\omega^{t-p q} \bar{A}_{q}^{\lambda}\right)_{1 \leqslant p, q \leqslant t-1}}\right) \tag{5.4.33}
\end{equation*}
$$

Substituting $M_{j}=B_{j}^{\lambda}, N_{j}=\bar{B}_{j}^{\lambda}, U_{j}=A_{j}^{\lambda}, V_{j}=-\bar{A}_{j}^{\lambda}$ and for $1 \leqslant j \leqslant t-1$,

$$
\gamma_{i, j}= \begin{cases}\omega^{\frac{i(j+1)}{2}} & j \text { odd } \\ \omega^{-\frac{i j}{2}} & j \text { even }\end{cases}
$$

in Lemma 5.27, we get the following corollary.
Corollary 5.33. 1. If

$$
n_{j}(\lambda, t n+1)+n_{t-j}(\lambda, t n+1)=\left\{\begin{array}{ll}
2 n+1+\delta_{i_{0}, \frac{t}{2}} & j=i_{0},  \tag{5.4.34}\\
2 n & \text { otherwise },
\end{array} \quad j \in[t-1],\right.
$$

for some $i_{0} \in\left[\left\lfloor\frac{t}{2}\right\rfloor\right]$, then

$$
\begin{equation*}
\operatorname{det} \Delta_{3}=(-1)^{\chi_{3}}(\operatorname{det} \Gamma)^{n} \operatorname{det}\left(\frac{O_{i_{0}}^{(3)}}{W_{i_{0}}^{(3)}}\right) \prod_{\substack{i=1 \\ i \neq i_{0}}}^{\left\lfloor\frac{t}{2}\right\rfloor} \operatorname{det} W_{i}^{(3)}, \tag{5.4.35}
\end{equation*}
$$

where

$$
\begin{gathered}
O_{i}^{(3)}=\left\{\begin{array}{lr}
\left(B_{i}^{\lambda}+\bar{B}_{i}^{\lambda} \mid B_{t-i}^{\lambda}+\bar{B}_{t-i}^{\lambda}\right) & \text { if } 1 \leqslant i \leqslant\left\lfloor\frac{t-1}{2}\right\rfloor, \\
\left(B_{\frac{t}{2}}^{\lambda}+\bar{B}_{\frac{t}{2}}^{\lambda}\right) & t \text { even and } i=\frac{t}{2},
\end{array}\right. \\
W_{i}^{(3)}= \begin{cases}\left(\begin{array}{c|}
A_{i}^{\lambda} \\
\hline \bar{A}_{t-i}^{\lambda} \\
\bar{A}_{i}^{\lambda} \\
A_{t-i}^{\lambda}
\end{array}\right) & 1 \leqslant i \leqslant\left\lfloor\frac{t-1}{2}\right\rfloor, \\
\left(A_{\frac{t}{2}}^{\lambda}+\bar{A}_{\frac{t}{2}}^{\lambda}\right) & t \text { even and } i=\frac{t}{2},\end{cases} \\
\text { and } \quad \chi_{3}=\left(\sum_{i=t+1-i_{0}}^{t-1} n_{i}(\lambda)\right)+\sum_{i=\left\lfloor\frac{t+2}{2}\right\rfloor}^{t-1}(t-1) n n_{i}(\lambda) .
\end{gathered}
$$

2. Otherwise, $\operatorname{det} \Delta_{3}=0$.

Proof of Theorem 5.10. By (2.4.5, we see that the numerator of the required even orthogonal character is:

$$
2 \operatorname{det}\left(\frac{\left(\left(\left(\omega^{p-1} x_{i}\right)^{\beta_{j}(\lambda)}+\left(\bar{\omega}^{p-1} \bar{x}_{i}\right)^{\beta_{j}(\lambda)}\right)_{1 \leqslant j \leqslant t n+1}^{1 \leqslant i \leqslant n}\right)_{1 \leqslant p \leqslant t}}{\left(y^{\beta_{j}(\lambda)}+\bar{y}^{\beta_{j}(\lambda)}\right)_{1 \leqslant j \leqslant t n+1}}\right)
$$

First permuting the columns of the matrix in the numerator by $\sigma_{\lambda}$ from (5.1.1) ( $\mathrm{m}=$ $1, d_{1}=0$ ) and then permuting the last $t$ rows cyclically, we see that the numerator is

$$
2 \operatorname{sgn}\left(\sigma_{\lambda}\right)(-1)^{(t-1)} \operatorname{det}\left(\begin{array}{c|c|c|c}
A_{0}^{\lambda}+\bar{A}_{0}^{\lambda} & A_{1}^{\lambda}+\bar{A}_{1}^{\lambda} & \ldots & A_{t-1}^{\lambda}+\bar{A}_{t-1}^{\lambda} \\
\hline B_{0}^{\lambda}+\bar{B}_{0}^{\lambda} & B_{1}^{\lambda}+\bar{B}_{1}^{\lambda} & \ldots & B_{t-1}^{\lambda}+\bar{B}_{t-1}^{\lambda} \\
\hline A_{0}^{\lambda}+\bar{A}_{0}^{\lambda} & \omega A_{1}^{\lambda}+\omega^{t-1} \bar{A}_{1}^{\lambda} & \ldots & \omega^{t-1} A_{t-1}^{\lambda}+\omega \bar{A}_{t-1}^{\lambda} \\
\hline \vdots & \vdots & \ldots & \vdots \\
A_{0}^{\lambda}+\bar{A}_{0}^{\lambda} & \omega^{t-1} A_{1}^{\lambda}+\omega \bar{A}_{1}^{\lambda} & \ldots & \omega A_{t-1}^{\lambda}+\omega^{t-1} \bar{A}_{t-1}^{\lambda},
\end{array}\right),
$$

where $A_{p, q}^{\lambda}, \bar{A}_{p}^{\lambda}, B_{p, q}^{\lambda}$ and $\bar{B}_{p}^{\lambda}$ are defined in 5.2.7) and (5.2.8). Applying blockwise row operations $R_{1} \rightarrow R_{1}+R_{3}+\cdots+R_{t+1}$ and then $R_{i} \rightarrow R_{i}-\frac{1}{t} R_{1}, 3 \leqslant i \leqslant t+1$, we get
$2 t^{n} \operatorname{sgn}\left(\sigma_{\lambda}\right)(-1)^{(t-1)} \operatorname{det}\left(\begin{array}{c|c|c|c}A_{0}^{\lambda}+\bar{A}_{0}^{\lambda} & 0 & \ldots & 0 \\ \hline B_{0}^{\lambda}+\bar{B}_{0}^{\lambda} & B_{1}^{\lambda}+\bar{B}_{1}^{\lambda} & \ldots & B_{t-1}^{\lambda}+\bar{B}_{t-1}^{\lambda} \\ \hline 0 & \omega A_{1}^{\lambda}+\omega^{t-1} \bar{A}_{1}^{\lambda} & \ldots & \omega^{t-1} A_{t-1}^{\lambda}+\omega \bar{A}_{t-1}^{\lambda} \\ \hline \vdots & \vdots & \ldots & \vdots \\ \hline 0 & \omega^{t-1} A_{1}^{\lambda}+\omega \bar{A}_{1}^{\lambda} & \ldots & \omega A_{t-1}^{\lambda}+\omega^{t-1} \bar{A}_{t-1}^{\lambda},\end{array}\right)$.

Since the denominator in 2.4.5) is the numerator evaluated at the empty partition and $n_{0}(\varnothing, t n+1)=n+1, n_{i}(\varnothing, t n+1)=n$ for all $i \in[1, t-1]$, evaluating the numerator in 5.4.36) and then using Corollary 5.32, we see that the denominator of the required even orthogonal character is:

$$
\begin{align*}
&\left(1+\delta_{t n+1}\right) t^{n} \operatorname{sgn}\left(\sigma_{\varnothing}\right)(-1)^{(t-1)+\Sigma_{3}^{0}}(\operatorname{det} \Gamma)^{n} \operatorname{det}\left(\frac{A_{0}+\bar{A}_{0}}{B_{0}+\bar{B}_{0}}\right) \\
& \times \prod_{q=1}^{\left\lfloor\frac{t-1}{2}\right\rfloor} \operatorname{det}\left(\begin{array}{l|l}
A_{q} & \bar{A}_{t-q} \\
\hline \bar{A}_{q} & A_{t-q}
\end{array}\right) \times \begin{cases}\operatorname{det}\left(A_{\frac{t}{2}}+\bar{A}_{\frac{t}{2}}\right) & t \text { even } \\
1 & t \text { odd }\end{cases} \tag{5.4.37}
\end{align*}
$$

where

$$
\Sigma_{3}^{0}= \begin{cases}\sum_{q=\frac{t+2}{2}}^{t-1} n^{2} & t \text { even } \\ 0 & t \text { odd }\end{cases}
$$

If $\operatorname{core}_{t}(\lambda) \notin \mathcal{Q}_{1,0,0}^{(t)} \cup \mathcal{Q}_{2,1, k}^{(t)}$ for all $k \in\left[\operatorname{rk}\left(\operatorname{core}_{t}(\lambda)\right)\right]$, then by Lemma 5.16. Corollary 5.19. Corollary 5.32 and Corollary 5.33, the numerator in (5.4.36) is 0 . So,

$$
\mathrm{o}_{\lambda}^{\text {even }}\left(X, \omega X, \ldots, \omega^{t-1} X, y\right)=0 .
$$

If $\operatorname{core}_{t}(\lambda) \in \mathcal{Q}_{1,0,0}^{(t)}$, then by Lemma 5.16. $n_{0}(\lambda)=n+1$ and $n_{i}(\lambda)+n_{t-i}(\lambda)=2 n$ and the numerator in (5.4.36) is the same as:

$$
\begin{equation*}
2 t^{n} \operatorname{sgn}\left(\sigma_{\lambda}\right)(-1)^{(t-1)} \operatorname{det}\left(\frac{A_{0}^{\lambda}+\bar{A}_{0}^{\lambda}}{B_{0}^{\lambda}+\bar{B}_{0}^{\lambda}}\right) \times \operatorname{det}\left(\omega^{i j} A_{j}^{\lambda}+\omega^{t-i j} \bar{A}_{j}^{\lambda}\right)_{1 \leqslant i, j \leqslant t-1} . \tag{5.4.38}
\end{equation*}
$$

The last matrix in (5.4.38) is $\Pi_{3}$ defined in (5.4.31). Using Corollary 5.32, we see that the numerator is

$$
\begin{align*}
2 t^{n} \operatorname{sgn}\left(\sigma_{\lambda}\right)(-1)^{(t-1)+\Sigma_{3}} & (\operatorname{det} \Gamma)^{n} \operatorname{det}\binom{A_{0}^{\lambda}+\bar{A}_{0}^{\lambda}}{\hline B_{0}^{\lambda}+\bar{B}_{0}^{\lambda}} \\
& \times \prod_{q=1}^{\left\lfloor\frac{t-1}{2}\right\rfloor} \operatorname{det}\left(\begin{array}{ll}
A_{q}^{\lambda} & \bar{A}_{t-q}^{\lambda} \\
\hline \bar{A}_{q}^{\lambda} & A_{t-q}^{\lambda}
\end{array}\right) \times \begin{cases}\operatorname{det}\left(A_{\frac{t}{2}}^{\lambda}+\bar{A}_{\frac{t}{2}}^{\lambda}\right) & t \text { even, } \\
1 & t \text { odd. }\end{cases} \tag{5.4.39}
\end{align*}
$$

By (5.2.14) and (2.4.8), we note that

$$
\frac{\operatorname{det}\left(A_{\frac{t}{2}}^{\lambda}+\bar{A}_{\frac{t}{2}}^{\lambda}\right)}{\operatorname{det}\left(A_{\frac{t}{2}}+\bar{A}_{\frac{t}{2}}\right)}=(-1)^{\sum_{i} \lambda_{i}^{(t / 2)}} \operatorname{so}_{\lambda(t / 2)}\left(-X^{t}\right) .
$$

So, applying (5.2.18) and Lemma 5.22, the ratio of (5.4.39) and (5.4.37), the required even orthogonal character is

$$
(-1)^{\epsilon_{3}} \operatorname{sgn}\left(\sigma_{\lambda}\right) \mathrm{o}_{\lambda(0)}^{\text {even }}\left(X^{t}, y^{t}\right) \prod_{i=1}^{\left\lfloor\frac{t-1}{2}\right\rfloor} s_{i}^{(3)}\left(X^{t}, \bar{X}^{t}\right) \times \begin{cases}(-1)^{\sum_{i} \lambda_{i}^{(t / 2)}} \operatorname{so}_{\lambda(t / 2)}\left(-X^{t}\right) & t \text { even }, \\ 1 & t \text { odd },\end{cases}
$$

where

$$
\left.\begin{array}{rl}
\epsilon_{3}=\frac{t(t-1)}{2} \frac{n(n+1)}{2}+ \begin{cases}n & \sum_{q=\frac{t+2}{2}}^{t-1}\left(n_{q}(\lambda)-n\right) \\
0 & t \text { even, }\end{cases}  \tag{5.4.40}\\
& \quad t \text { odd }
\end{array}\right\}
$$

If $n_{0}(\lambda)=n$, then the numerator is

$$
2 t^{n} \operatorname{sgn}\left(\sigma_{\lambda}\right)(-1)^{(t-1)} \operatorname{det}\left(A_{0}^{\lambda}+\bar{A}_{0}^{\lambda}\right) \times \operatorname{det}\left(\frac{\left(B_{j}+\bar{B}_{j}\right)_{1 \leqslant j \leqslant t-1}}{\left(\omega^{i j} A_{j}^{\lambda}+\omega^{t-i j} \bar{A}_{j}^{\lambda}\right)_{1 \leqslant i, j \leqslant t-1}}\right)
$$

The last matrix is the same as $\Delta_{3}$ defined in (5.4.33). We use Corollary 5.33 to factorize the determinant. If $\operatorname{core}_{t}(\lambda) \in \mathcal{Q}_{2,1, k}^{(t)}$ for all $k \in\left[\operatorname{rk}\left(\operatorname{core}_{t}(\lambda)\right)\right]$, then by Corollary 5.19 (5.4.34) holds. Case 1. $t$ is even and $i_{0}=\frac{t}{2}$. Then $n_{t / 2}(\lambda)=n+1$ and using Corollary 5.33 , the factorization for the numerator is

$$
2 t^{n} \operatorname{sgn}\left(\sigma_{\lambda}\right)(-1)^{(t-1)+\chi_{3}}(\operatorname{det} \Gamma)^{n} \operatorname{det}\left(A_{0}^{\lambda}+\bar{A}_{0}^{\lambda}\right) \operatorname{det}\binom{B_{\frac{t}{2}}^{\lambda}+\bar{B}_{\frac{t}{2}}^{\lambda}}{\hline A_{\frac{t}{2}}^{\lambda}+\bar{A}_{\frac{t}{2}}^{\lambda}} \prod_{q=1}^{\frac{t-2}{2}} \operatorname{det}\left(\begin{array}{c|c}
A_{q}^{\lambda} & \bar{A}_{t-q}^{\lambda}  \tag{5.4.41}\\
\hline \bar{A}_{q}^{\lambda} & A_{t-q}^{\lambda}
\end{array}\right) .
$$

By Lemma 5.24, we have

$$
\begin{equation*}
\operatorname{det}\left(A_{\frac{t}{2}}+\bar{A}_{\frac{t}{2}}\right)=\frac{1}{2} \operatorname{det}\left(x_{i}^{t(n-j)}+\bar{x}_{i}^{t(n-j)}\right) \prod_{i=1}^{n}\left(x_{i}^{t / 2}+\bar{x}_{i}^{t / 2}\right) \tag{5.4.42}
\end{equation*}
$$

Substituting in (5.4.37), the denominator is

$$
\begin{align*}
\left(1+\delta_{t n+1}\right) t^{n} & \operatorname{sgn}\left(\sigma_{\varnothing}\right)(-1)^{(t-1)+\Sigma_{3}^{0}}(\operatorname{det} \Gamma)^{n} \operatorname{det}\left(\frac{A_{0}+\bar{A}_{0}}{B_{0}+\bar{B}_{0}}\right) \\
& \times \prod_{q=1}^{\left\lfloor\frac{t-1}{2}\right\rfloor} \operatorname{det}\left(\begin{array}{c|c}
A_{q} & \bar{A}_{t-q} \\
\hline \bar{A}_{q} & A_{t-q}
\end{array}\right) \times \frac{1}{2} \operatorname{det}\left(x_{i}^{t(n-j)}+\bar{x}_{i}^{t(n-j)}\right) \prod_{i=1}^{n}\left(x_{i}^{t / 2}+\bar{x}_{i}^{t / 2}\right) \tag{5.4.43}
\end{align*}
$$

By (5.2.14) and (2.4.8), we note that

$$
\frac{\operatorname{det}\binom{B_{\frac{t}{2}}^{\lambda}+\bar{B}_{\frac{t}{2}}^{\lambda}}{A_{\frac{t}{2}}^{\lambda}+\bar{A}_{\frac{t}{2}}^{\lambda}}}{\prod_{i=1}^{n}\left(x_{i}^{t / 2}+\bar{x}_{i}^{t / 2}\right) \operatorname{det}\left(\frac{B_{0}+\bar{B}_{0}}{A_{0}+\bar{A}_{0}}\right)}=(-1)^{\Sigma_{i} \lambda_{i}^{(t / 2)}}\left(y^{t / 2}+\bar{y}^{t / 2}\right) \operatorname{so}_{\lambda(t / 2)}\left(-X^{t},-y^{t}\right)
$$

Note that $\lambda_{t n+1}$ is zero iff $\lambda_{n}^{(0)}$. Hence, using (5.2.14), Lemma 5.22 and the ratio of (5.4.41) and (5.4.43), the even orthogonal character is

$$
(-1)^{\epsilon_{3}+n} \operatorname{sgn}\left(\sigma_{\lambda}\right)\left(y^{t / 2}+\bar{y}^{t / 2}\right) o_{\lambda(0)}^{\operatorname{even}}\left(X^{t}\right)(-1)^{\sum_{i} \lambda_{i}^{(t / 2)}} \mathrm{so}_{\lambda(t / 2)}\left(-X^{t},-y^{t}\right) \prod_{q=1}^{\frac{t-2}{2}} s_{\pi_{q}^{(1)}}\left(X^{t}, \bar{X}^{t}\right)
$$

where

$$
\begin{aligned}
\epsilon_{3}=\frac{t(t-1)}{2} \frac{n(n+1)}{2}+\left(\sum_{i=t+1-i_{0}}^{t-1} n_{i}(\lambda)\right) & +\sum_{i=\left\lfloor\frac{t+2}{2}\right\rfloor}^{t-1}(t-1) n\left(n_{i}(\lambda)-n\right) \\
& +\sum_{q=1}^{\frac{t-2}{2}}\left(\frac{n_{t-q}(\lambda)\left(n_{t-q}(\lambda)-1\right)}{2}-\frac{n(n-1)}{2}\right) .
\end{aligned}
$$

Case 2. $i_{0} \neq \frac{t}{2}$. In this case, the factorization for the numerator is

$$
\begin{aligned}
& 2 t^{n} \operatorname{sgn}\left(\sigma_{\lambda}\right)(-1)^{(t-1)+\chi_{3}}(\operatorname{det} \Gamma)^{n} \operatorname{det}\left(A_{0}^{\lambda}+\bar{A}_{0}^{\lambda}\right) \operatorname{det}\left(\begin{array}{c|c}
B_{i_{0}}^{\lambda}+\bar{B}_{i_{0}}^{\lambda} & B_{t-i_{0}}^{\lambda}+\bar{B}_{t-i_{0}}^{\lambda} \\
\hline A_{i_{0}}^{\lambda} & \bar{A}_{t-i_{0}}^{\lambda} \\
\hline \bar{A}_{t-i_{0}}^{\lambda} & A_{i_{0}}^{\lambda}
\end{array}\right)_{2 n+1 \times 2 n+1} \\
& \times \prod_{\substack{q=1 \\
q \neq i_{0}}}^{\left\lfloor\left\lfloor\frac{t-1}{2}\right\rfloor\right.} \operatorname{det}\left(\begin{array}{ll}
A_{q} & \bar{A}_{t-q} \\
\hline \bar{A}_{q} & A_{t-q}
\end{array}\right) \times \begin{cases}\operatorname{det}\left(A_{t / 2}^{\lambda}+\bar{A}_{t / 2}^{\lambda}\right) & t \text { is even, } \\
1 & t \text { is odd. }\end{cases}
\end{aligned}
$$

By Lemma 5.24, we have

$$
\begin{gathered}
\operatorname{det}\left(\frac{A_{0}+\bar{A}_{0}}{B_{0}+\bar{B}_{0}}\right)=(-1)^{n} y^{-n} \prod_{i=1}^{n}\left(y-x_{i}\right)\left(y-\bar{x}_{i}\right) \operatorname{det}\left(x_{i}^{n-j}+\bar{x}_{i}^{n-j}\right), \\
\operatorname{det}\left(A_{\frac{t}{2}}+\bar{A}_{\frac{t}{2}}\right)=\frac{1}{2} \prod_{i=1}^{n}\left(x_{i}^{t / 2}+\bar{x}_{i}^{t / 2}\right) \operatorname{det}\left(x_{i}^{n-j}+\bar{x}_{i}^{n-j}\right), \\
\operatorname{det}\left(\begin{array}{c|c}
A_{i_{0}} & \bar{A}_{t-i_{0}} \\
\hline \bar{A}_{i_{0}} & A_{t-i_{0}}
\end{array}\right)=(-1)^{\frac{n(n-1)}{2}} \prod_{1 \leqslant i<j \leqslant n}\left(x_{i}^{t}-x_{j}^{t}\right)\left(x_{i}^{t}-\bar{x}_{j}^{t}\right)\left(x_{j}^{t}-\bar{x}_{i}^{t}\right)\left(\bar{x}_{i}^{t}-\bar{x}_{j}^{t}\right) \prod_{i=1}^{n}\left(x_{i}^{t}-\bar{x}_{i}^{t}\right) .
\end{gathered}
$$

In this case, using Lemma 5.23 and Lemma 5.22, the required even orthogonal character is

$$
\begin{aligned}
& (-1)^{\epsilon 3} \operatorname{sgn}\left(\sigma_{\lambda}\right) \mathrm{o}_{\lambda(0)}^{\mathrm{even}}\left(X^{t}\right)\left(y^{-t\left(\lambda_{1}^{\left(t-i_{0}\right)}+n_{\left(t-i_{0}\right)}(\lambda)-n\right)+i_{0}} s_{\pi_{i_{0}}^{(3)}}\left(X^{t}, \bar{X}^{t}, y^{t}\right)+y^{t\left(\lambda_{1}^{\left(t-i_{0}\right)}+n_{t-i_{0}}(\lambda)+n\right)-i_{0}}\right. \\
& \left.\quad \times s_{\pi_{i_{0}}^{(3)}}\left(X^{t}, \bar{X}^{t}, \bar{y}^{t}\right)\right) \times \prod_{\substack{j=1 \\
j \neq i_{0}}}^{\left\lfloor\frac{t-1}{2}\right\rfloor} s_{\pi_{i}^{(3)}}\left(X^{t}, \bar{X}^{t}\right) \times \begin{cases}(-1)_{i} \lambda_{i}^{(t / 2)} \operatorname{so}_{\lambda(t / 2)}\left(-X^{t}\right) & t \text { is even }, \\
1 & t \text { is odd. }\end{cases}
\end{aligned}
$$

This completes the proof.

### 5.5 Generating functions

We now give enumerative results for $\left(z_{1}, z_{2}, k\right)$-asymmetric partitions defined in (5.3).
Proposition 5.34. Fix $z_{1}>z_{2} \geqslant 0$ and $k \geqslant 1$. The number of $\left(z_{1}, z_{2}, k\right)$-asymmetric partitions of $m$ is equal to the number of partitions of $m$ of the form

$$
\begin{align*}
(z_{1}(r-1)+r, \underbrace{2 r-1, \ldots, 2 r-1}_{\alpha_{r}}, & , \underbrace{2 r-3, \ldots, 2 r-3}_{\alpha_{r-1}-\alpha_{r}}, \ldots, \underbrace{2 k-1, \ldots, 2 k-1}_{\alpha_{k}-\alpha_{k+1}} \\
& \underbrace{2 k-2, \ldots, 2 k-2}_{\alpha_{k-1}-\alpha_{k}}, \ldots, \underbrace{2, \ldots, 2}_{\alpha_{1}-\alpha_{2}}, \underbrace{1, \ldots, 1}_{z_{2}}) \tag{5.5.1}
\end{align*}
$$

for $r \geqslant 1$ and $\left\{\alpha_{1}, \ldots, \alpha_{r}\right\}_{>} \subset \mathbb{Z}_{\geqslant 0}$.
Proof. Let $\lambda=\left(\alpha_{1}, \ldots, \alpha_{k}, \ldots, \alpha_{r} \mid \alpha_{1}+z_{1}, \ldots, \widehat{\alpha_{k}+z_{1}}, \ldots, \alpha_{r}+z_{1}, z_{2}\right)$ be a $\left(z_{1}, z_{2}, k\right)$ asymmetric partition of $m$. It is easy to see that the mapping $\lambda$ to the partition in (5.5.1) gives the required bijection.

Proposition 5.34 gives an expression for the generating function:

## Corollary 5.35.

$$
\sum_{\lambda \in \mathcal{Q}_{z_{1}, z_{2}, k}} q^{|\lambda|}=\sum_{n \geqslant 1} \frac{q^{\left(z_{1}+n\right)(n-1)+z_{2}+k}}{\left(1-q^{2}\right) \cdots\left(1-q^{2 k-2}\right)\left(1-q^{2 k-1}\right) \cdots\left(1-q^{2 n-1}\right)} .
$$

Recall, $\mathcal{Q}_{z_{1}, z_{2}, k}^{(t)}$ from Definition 5.3. For $z_{1}>z_{2}$, let

$$
\mathcal{Q}_{z_{1}, z_{2}}^{(t)}=\bigcup_{k} \mathcal{Q}_{z_{1}, z_{2}, k}^{(t)} .
$$

We now enumerate the $t$-core partitions in $\mathcal{Q}_{z+2,0}^{(t)} \cup \mathcal{Q}_{z+2, z+1}^{(t)}$. Represent the elements of $\mathbb{Z}^{\left\lfloor\frac{t-z}{2}\right\rfloor} \times\left\{0, \ldots,\left\lfloor\frac{t-z-1}{2}\right\rfloor, t-z, \ldots, t-1\right\}$ by $(\vec{v}, \check{v}):=\left(v_{0}, \ldots, v_{\left\lfloor\frac{t-z-2}{2}\right\rfloor}, \check{v}\right)$.

Theorem 5.36. Fix $0<z+2 \leqslant t+2$. Define $\vec{b} \in \mathbb{Z}^{\left\lfloor\frac{t-z}{2}\right\rfloor}$ by $\vec{b}_{i}:=t-z-1-2 i$. Then there exists a bijection $\psi: \mathcal{Q}_{z+2,0}^{(t)} \cup \mathcal{Q}_{z+2, z+1}^{(t)} \rightarrow \mathbb{Z}^{\left.\frac{t-z}{2}\right\rfloor} \times\left\{0, \ldots,\left\lfloor\frac{t-z-1}{2}\right\rfloor, t-z, \ldots, t-1\right\}$ satisfying

$$
|\lambda|=t \| \psi\left(\overrightarrow{(\lambda)} \|^{2}-\vec{b} \cdot \psi\left(\overrightarrow{(\lambda)}+ \begin{cases}\overline{\psi(\lambda)} & \overline{\psi(\lambda)} \in[t-z, t-1] \cup\left\{\frac{t-z-1}{2}\right\}, \\ t\left(n_{\overline{\psi(\lambda)}}(\lambda)-n\right)+t-z-1 & \text { otherwise },\end{cases}\right.\right.
$$

where • represents the standard inner product.
Proof. Suppose $\lambda \in \mathcal{Q}_{z+2,0}^{(t)} \cup \mathcal{Q}_{z+2, z+1}^{(t)}$ such that $\ell(\lambda) \leqslant t n+1$ for some $n \geqslant 1$. Then by Lemma 5.18, there exists a unique $i_{0} \in\left[0,\left[\frac{t-z-1}{2}\right]\right] \cup[t-z, t-1]$ such that 5.2.3) holds.

Define the map $\psi$ by

$$
(\psi(\lambda))_{i}:=n_{i}(\lambda)-n, \quad 0 \leqslant i \leqslant\left\lfloor\frac{t-z-2}{2}\right\rfloor, \quad \overline{\psi(\lambda)}:=i_{0} .
$$

Since $n$ is not unique, it is not a priori clear that $\psi$ is well-defined. But from the definition of $n_{i}(\lambda)$ ), it is easy to see that $n_{i}(\lambda, t n+1)-n=n_{i}(\lambda, t n+t+1)-n-1$. Hence, $\psi(\lambda)$ is indeed well-defined.

To show that $\psi$ is a bijection, we define the inverse of $\psi$ as follows. For a vector $(\vec{v}, \check{v})=\left(v_{0}, \ldots, v_{\left\lfloor\frac{t-z-2}{2}\right\rfloor}, \check{v}\right)$, let $n=\max \left\{\left|v_{0}\right|,\left|v_{1}\right|, \ldots,\left|v_{\left\lfloor\frac{t-z-2}{2}\right\rfloor}\right|\right\}$ and for $0 \leqslant i \leqslant t-1$,

$$
m_{i}=\left\{\begin{array}{ll}
n+v_{i} & 0 \leqslant i \leqslant\left\lfloor\frac{t-z-2}{2}\right\rfloor \\
n-v_{t-z-1-i} & \left\lfloor\frac{t-z+1}{2}\right\rfloor \leqslant i \leqslant t-z-1, \\
n & \text { otherwise }
\end{array} \text { and } r_{i}= \begin{cases}m_{i}+1 & i=\check{v} \\
m_{i} & \text { otherwise }\end{cases}\right.
$$

By construction, $\sum_{i=0}^{t-1} r_{i}=t n+1$,

$$
\begin{aligned}
r_{i}+r_{t-z-1-i} & =\left\{\begin{array}{ll}
2 n+1+\delta_{\check{v}, \frac{t-z-1}{2}} & \text { if } i=\check{v} \\
2 n & \text { otherwise }
\end{array} \quad \text { for } \quad 0 \leqslant i \leqslant\left\lfloor\frac{t-z-1}{2}\right\rfloor,\right. \\
\text { and } \quad r_{i} & =\left\{\begin{array}{lll}
n+1 & \text { if } i=\check{v} \\
n & \text { otherwise }
\end{array} \text { for } t-z \leqslant i \leqslant t-1,\right.
\end{aligned}
$$

By Lemma 5.18, there is a unique $t$-core $\lambda \in \mathcal{Q}_{z+2,0}^{(t)} \cup \mathcal{Q}_{z+2, z+1}^{(t)}$ satisfying $n_{i}(\lambda)=r_{i}$. and we set $\psi^{-1}(\vec{v}, \check{v})=\lambda$. Moreover the size of $\lambda$ is computed as

$$
\begin{equation*}
|\lambda|=\sum_{i=1}^{t n+1} \beta_{i}(\lambda)-\frac{t n(t n+1)}{2} . \tag{5.5.2}
\end{equation*}
$$

Since $\lambda$ is a $t$-core, $t j+i, 0 \leqslant j \leqslant n_{i}(\lambda)-1,0 \leqslant i \leqslant t-1$ are the parts of $\beta(\lambda)$ (see Proposition 2.3). So,

$$
\begin{aligned}
\sum_{i=1}^{t n+1} \beta_{i}(\lambda)=\sum_{i=0}^{t-1}\left(i\left(n_{i}(\lambda)\right)\right. & \left.+\frac{n_{i}(\lambda)\left(n_{i}(\lambda)-1\right) t}{2}\right) \\
& =\sum_{i=0}^{t-1}\left(i\left(n_{i}(\lambda)-n\right)\right)+\frac{t n(t-1)}{2}+\frac{t}{2} \sum_{i=0}^{t-1} n_{i}(\lambda)^{2}-\frac{t(t n+1)}{2} .
\end{aligned}
$$

Substituting this in (5.5.2) gives

$$
\begin{aligned}
|\lambda|=\sum_{i=0}^{t-1}\left(i\left(n_{i}(\lambda)-n\right)\right)+\frac{t}{2}\left(\sum_{i=0}^{t-1} n_{i}(\lambda)^{2}\right. & \left.-t n^{2}-2 n-1\right) \\
& =\sum_{i=0}^{t-1}\left(i\left(n_{i}(\lambda)-n\right)\right)+\frac{t}{2} \sum_{i=0}^{t-1}\left(n_{i}(\lambda)-n\right)^{2}-\frac{t}{2} .
\end{aligned}
$$

Since $\lambda \in \mathcal{Q}_{z+2,0}^{(t)} \cup \mathcal{Q}_{z+2, z+1}^{(t)}$, using Lemma 5.18, we have

$$
\sum_{i=0}^{t-1}\left(n_{i}(\lambda)-n\right)^{2}=2\|\vec{v}\|^{2}+1- \begin{cases}0 & \check{v} \in[t-z, t-1] \cup\left\{\frac{t-z-1}{2}\right\} \\ 2\left(n_{\check{v}}(\lambda)-n\right) & \text { otherwise }\end{cases}
$$

and

$$
\sum_{i=0}^{t-1} i\left(n_{i}(\lambda)-n\right)=\sum_{i=0}^{\left\lfloor\frac{t-z-2}{2}\right\rfloor}(2 i+z+1-t)\left(n_{i}(\lambda)-n\right)+ \begin{cases}\check{v} & \check{v} \in[t-z, t-1] \cup\left\{\frac{t-z-1}{2}\right\} \\ t-z-1 & \text { otherwise }\end{cases}
$$

Now observe that

$$
\begin{gathered}
-\vec{b} \cdot \vec{v}=\sum_{i=0}^{\left.\frac{t-z-2}{2}\right\rfloor}(2 i+z+1-t)\left(n_{i}(\lambda)-n\right) \\
-\vec{b} \cdot \vec{v}=\sum_{i=0}^{t-1} i\left(n_{i}(\lambda)-n\right)- \begin{cases}\check{v} & \check{v} \in[t-z, t-1] \cup\left\{\frac{t-z-1}{2}\right\}, \\
t-z-1 & \text { otherwise. }\end{cases} \\
\frac{t}{2} \sum_{i=0}^{t-1}\left(n_{i}(\lambda)-n\right)^{2}-\frac{t}{2}=t\|\vec{v}\|^{2}- \begin{cases}0 & \check{v} \in[t-z, t-1] \cup\left\{\frac{t-z-1}{2}\right\}, \\
t\left(n_{\check{v}}(\lambda)-n\right) & \text { otherwise. }\end{cases}
\end{gathered}
$$

Hence

$$
|\lambda|=t\|\vec{v}\|^{2}-\vec{b} \cdot \vec{v}+ \begin{cases}\check{v} & \check{v} \in[t-z, t-1] \cup\left\{\frac{t-z-1}{2}\right\}, \\ t\left(n_{\check{v}}(\lambda)-n\right)+t-z-1 & \text { otherwise }\end{cases}
$$

completing the proof.
Corollary 5.37. There are infinitely many $t$-cores in $\mathcal{Q}_{z+2,0}^{(t)} \cup \mathcal{Q}_{z+2, z+1}^{(t)}$ for $t \geqslant z$.

## Chapter 6

## Skew hook Schur functions and the cyclic sieving phenomenon

In Section 6.1 of this chapter, we consider specialized skew hook Schur (supersymmmetric skew Schur) polynomial $\mathrm{hs}_{\lambda / \mu}\left(X, \omega X, \ldots, \omega^{t-1} X / Y, \omega Y, \ldots, \omega^{t-1} Y\right)$, where $\omega^{k} X=$ $\left(\omega^{k} x_{1}, \ldots, \omega^{k} x_{n}\right), \omega^{k} Y=\left(\omega^{k} y_{1}, \ldots, \omega^{k} y_{m}\right)$ for $0 \leqslant k \leqslant t-1$ and give a combinatorial interpretation of $\left.\mathrm{hs}_{\lambda / \mu}\left(1, \omega^{d}, \ldots, \omega^{d(t n-1)} / 1, \omega^{d}, \ldots, \omega^{d(t m-1}\right)\right)$, for all divisors $d$ of $t$, in terms of ribbon supertableaux. Then we give a combinatorial proof of the skew Schur factorization result in Section 6.2. Furthermore, in Section 6.3 we use the combinatorial interpretation to prove the cyclic sieving phenomenon on the set of semistandard supertableaux of shape $\lambda / \mu$ for odd $t$, and using a similar proof strategy, we give a complete generalization of a result of Lee-Oh [73] for the cyclic sieving phenomenon on the set of skew SSYT conjectured by Alexandersson-Pfannerer-Rubey-Uhlin [6]. A preprint of this work has appeared on arXiv [71].

### 6.1 Skew hook Schur polynomial factorization

In this section, we consider the specialized skew hook Schur polynomials. Recall, $X=$ $\left(x_{1}, \ldots, x_{n}\right), Y=\left(y_{1}, \ldots, y_{m}\right)$ and $\omega$ is a primitive $t^{\prime}$ th root of unity. We denote our indeterminates by $\left(X^{(\omega)} / Y^{(\omega)}\right):=\left(X, \omega X, \omega^{2} X, \ldots, \omega^{t-1} X / Y, \omega Y, \omega^{2} Y, \ldots, \omega^{t-1} Y\right)$.

Theorem 6.1. For $k \geqslant 0$, the complete supersymmetric function $H_{k}\left(X^{(\omega)} / Y^{(\omega)}\right)$ is given by

$$
H_{k}\left(X^{(\omega)} / Y^{(\omega)}\right)= \begin{cases}0 & \text { if } k \not \equiv 0 \quad(\bmod t) \\ H_{\frac{k}{t}}\left(X^{t} /(-1)^{t-1} Y^{t}\right) & \text { otherwise } .\end{cases}
$$

Proof. By (2.5.2, the required complete supersymmetric function is

$$
\begin{equation*}
H_{k}\left(X^{(\omega)} / Y^{(\omega)}\right)=\sum_{l=0}^{k} h_{l}\left(X^{(\omega)}\right) e_{k-l}\left(Y^{(\omega)}\right) \tag{6.1.1}
\end{equation*}
$$

By the generating function identities in (2.2.5), we have

$$
\begin{aligned}
\sum_{r \geqslant 0} e_{r}\left(Y^{(\omega)}\right) q^{r}=\prod_{i=1}^{m} & \left(1+y_{i} q\right)\left(1+y_{i} \omega q\right) \ldots\left(1+y_{i} \omega^{t-1} q\right) \\
& =\prod_{i=1}^{m}\left(1+\omega^{\frac{t(t-1)}{2}} y_{i}^{t} q^{t}\right)=\sum_{b \geqslant 0} e_{b}\left((-1)^{t-1} Y^{t}\right) q^{b t}
\end{aligned}
$$

and

$$
\sum_{r \geqslant 0} h_{r}\left(X^{(\omega)}\right) q^{r}=\prod_{i=1}^{n} \frac{1}{\left(1-x_{i} q\right)\left(1-x_{i} \omega q\right) \ldots\left(1-x_{i} \omega^{t-1} q\right)}=\sum_{b \geqslant 0} h_{b}\left(X^{t}\right) q^{b t}
$$

On comparing the coefficients, we see that $e_{r}\left(Y^{(\omega)}\right)$ and $h_{r}\left(X^{(\omega)}\right)$ are nonzero if and only if $t$ divides $r$. In that case

$$
\begin{equation*}
e_{r}\left(Y^{(\omega)}\right)=e_{\frac{r}{t}}\left((-1)^{t-1} Y^{t}\right) \quad \text { and } \quad h_{r}\left(X^{(\omega)}\right)=h_{\frac{r}{t}}\left(X^{t}\right) . \tag{6.1.2}
\end{equation*}
$$

If $t$ does not divide $k$, then for each $l \in[0, k]$, either $h_{l}\left(X^{(\omega)}\right)$ or $e_{k-l}\left(Y^{(\omega)}\right)$ is zero. This implies $H_{k}\left(X^{(\omega)} / Y^{(\omega)}\right)=0$. And if $t$ divides $k$, then substituting the values 6.1.2 in (6.1.1), we have

$$
H_{k}\left(X^{(\omega)} / Y^{(\omega)}\right)=\sum_{l=0}^{\frac{k}{t}} h_{l}\left(X^{t}\right) e_{\frac{k}{t}-l}\left((-1)^{t-1} Y^{t}\right)=H_{\frac{k}{t}}\left(X^{t} /(-1)^{t-1} Y^{t}\right)
$$

This completes the proof.
For a partition of length at most $t n$, let $\sigma_{\lambda} \in S_{t n}$ be the permutation that rearranges the parts of $\beta(\lambda)$ such that

$$
\begin{equation*}
\beta_{\sigma_{\lambda}(j)}(\lambda) \equiv q \quad(\bmod t), \quad \sum_{i=0}^{q-1} n_{i}(\lambda)+1 \leqslant j \leqslant \sum_{i=0}^{q} n_{i}(\lambda) \tag{6.1.3}
\end{equation*}
$$

arranged in decreasing order for each $q \in\{0,1, \ldots, t-1\}$. For the empty partition, $\beta(\varnothing, t n)=(t n-1, t n-2, \ldots, 0)$ with $n_{q}(\varnothing, t n)=n, 0 \leqslant q \leqslant t-1$ and

$$
\begin{equation*}
\sigma_{\varnothing}=(t, \ldots, n t, t-1, \ldots, n t-1, \ldots, 1, \ldots,(n-1) t+1) \tag{6.1.4}
\end{equation*}
$$

in one line notation with $\operatorname{sgn}\left(\sigma_{\varnothing}\right)=(-1)^{\frac{t(t-1)}{2} \frac{n(n+1)}{2}}$.
Lemma 6.2 ([77, Chapter I.1, Example 8(a)]). Let $\lambda, \mu$ be partitions of length at most $\ell$ such that $\mu \subset \lambda$, and such that $\lambda / \mu$ is a border strip of length $t$. Then $\beta(\mu)$ can be obtained from $\beta(\lambda)$ by subtracting $t$ from some part $\beta_{i}(\lambda)$ and rearranging in descending order.

Lemma 6.3. Let $\lambda$ and $\mu$ be partitions of length at most tn such that $\operatorname{core}_{t}(\lambda) / \operatorname{core}_{t}(\mu)$ is empty. Then

$$
\operatorname{sgn}\left(\sigma_{\lambda}\right) \operatorname{sgn}\left(\sigma_{\mu}\right)=(-1)^{\sum_{B \in \operatorname{Rib}(\lambda / \mu)} \operatorname{ht}(B)}
$$

Proof. Suppose $\mu$ is obtained from $\lambda$ by removing a border strip $\xi$. By Lemma 6.2, we see that $\beta(\mu)$ can be obtained from $\beta(\lambda)$ by subtracting $t$ from some part $\beta_{i}(\lambda)$ and rearranging in descending order. The height of $\xi$ is precisely the number of shifted transpositions we applied. Proceeding inductively completes the proof.

By Proposition 2.3, one can easily see that the Remark 6.4 holds true.
Remark 6.4. For partitions $\lambda$ and $\mu$ of length at most $t n$, $\operatorname{core}_{t}(\lambda) / \operatorname{core}_{t}(\mu)$ is empty if and only if $n_{i}(\lambda)=n_{i}(\mu)$ for all $i \in[0, t-1]$.

Theorem 6.5. Let $\lambda$ and $\mu$ be partitions of length at most tn. Then the skew hook Schur polynomial $\mathrm{hs}_{\lambda / \mu}\left(X^{(\omega)} / Y^{(\omega)}\right)$ is given by

1. If $\operatorname{core}_{t}(\lambda) / \operatorname{core}_{t}(\mu)$ is non-empty, then

$$
\mathrm{hs}_{\lambda / \mu}\left(X^{(\omega)} / Y^{(\omega)}\right)=0 .
$$

2. If $\operatorname{core}_{t}(\lambda) / \operatorname{core}_{t}(\mu)$ is empty, then

$$
\mathrm{hs}_{\lambda / \mu}\left(X^{(\omega)} / Y^{(\omega)}\right)=\operatorname{sgn}\left(\sigma_{\lambda}\right) \operatorname{sgn}\left(\sigma_{\mu}\right) \prod_{i=0}^{t-1} \mathrm{hs}_{\lambda^{(i)} / \mu^{(i)}}\left(X^{t} /(-1)^{t-1} Y^{t}\right)
$$

Proof. By the Jacobi-Trudi type identity (2.5.6) for the skew hook Schur polynomials, we see that the required skew hook Schur polynomial is

$$
\begin{equation*}
\mathrm{hs}_{\lambda / \mu}\left(X^{(\omega)} / Y^{(\omega)}\right)=\operatorname{det}\left(H_{\lambda_{i}-\mu_{j}-i+j}\left(X^{(\omega)} / Y^{(\omega)}\right)\right)=\operatorname{det}\left(H_{\beta_{i}(\lambda)-\beta_{j}(\mu)}\left(X^{(\omega)} / Y^{(\omega)}\right) .\right. \tag{6.1.5}
\end{equation*}
$$

Permuting the rows and columns of the determinant by $\sigma_{\lambda}$ and $\sigma_{\mu}$ respectively, defined in (6.1.3), we see that the skew hook Schur polynomial is

$$
\operatorname{sgn}\left(\sigma_{\lambda}\right) \operatorname{sgn}\left(\sigma_{\mu}\right) \operatorname{det}_{1 \leqslant i, j \leqslant t n}\left(H_{\beta_{\sigma_{\lambda}(i)}(\lambda)-\beta_{\sigma_{\mu}(j)}(\mu)}\right),
$$

where $H_{j}=H_{j}\left(X^{(\omega)} / Y^{(\omega)}\right)$ for all $j \in \mathbb{Z}$. By Theorem 6.1, we have $H_{j}=0$ if $j \equiv 0$ $(\bmod t)$ and $j>0$. Also, $H_{j}=0$ if $j<0$. Substituting these values in the above determinant, we see that the skew hook Schur polynomial is

$$
\operatorname{sgn}\left(\sigma_{\lambda}\right) \operatorname{sgn}\left(\sigma_{\mu}\right) \operatorname{det}\left(\begin{array}{cccc}
\widehat{S}_{0} & & & \\
& \widehat{S}_{1} & & 0 \\
& & \ddots & \\
0 & & & \widehat{S}_{t-1}
\end{array}\right)
$$

where

$$
\widehat{S}_{p}=\left(H_{\beta_{i}^{(p)}(\lambda)-\beta_{j}^{(p)}(\mu)}\right)_{\substack{1 \leqslant i \leqslant n_{p}(\lambda) \\ 1 \leqslant j \leqslant n_{p}(\mu)}} p \in[0, t-1] .
$$

If $\operatorname{core}_{t}(\lambda) / \operatorname{core}_{t}(\mu)$ is non-empty, then by Remark 6.4, $n_{i}(\lambda) \neq n_{i}(\mu)$ for some $i \in[0, t-1]$, then the $(i+1)^{\text {th }}$ diagonal block is not a square block. So, $\mathrm{hs}_{\lambda / \mu}\left(X^{(\omega)} / Y^{(\omega)}\right)=0$. If $\operatorname{core}_{t}(\lambda) / \operatorname{core}_{t}(\mu)$ is empty, then again by Remark $6.4, n_{i}(\lambda)=n_{i}(\mu)$ for all $0 \leqslant i \leqslant t-1$. Finally, by Proposition 2.3 and 2.5.6),

$$
\operatorname{det}\left(\widehat{S}_{p}\right)=\mathrm{hs}_{\lambda^{(p)}}\left(X^{t} /(-1)^{(t-1)} Y^{t}\right)
$$

we get the desired result.
Since hs ${ }_{\lambda / \mu}(X / \varnothing)=s_{\lambda / \mu}(X)$, we have the following corollary.
Corollary 6.6. Let $\lambda$ and $\mu$ be partitions of length at most tn. Then the skew Schur polynomial $s_{\lambda / \mu}\left(X^{(\omega)}\right)$ is given by

1. If $\operatorname{core}_{t}(\lambda) / \operatorname{core}_{t}(\mu)$ is non-empty, then

$$
s_{\lambda / \mu}\left(X^{(\omega)}\right)=0
$$

2. If $\operatorname{core}_{t}(\lambda) / \operatorname{core}_{t}(\mu)$ is empty, then

$$
s_{\lambda / \mu}\left(X^{(\omega)}\right)=\operatorname{sgn}\left(\sigma_{\lambda}\right) \operatorname{sgn}\left(\sigma_{\mu}\right) \prod_{i=0}^{t-1} s_{\lambda^{(i)} / \mu^{(i)}}\left(X^{t}\right)
$$

A generalization of Corollary 6.6 to skew characters was discovered by Farahat [36]. He gave an algebraic proof stated in an alternative language of star diagrams. In addition, a character-theoretic proof is given by Kerber, Sänger, and Wagner in [59]. Furthermore, Evseev, Paget and Wildon prove the result bijectively [35]. Also, Theorem 6.5 can be
derived from Corollary 6.6 using the notion of plethystic difference of variables $X$ and $Y$. See [22, 46, 72] for background on plethysm and plethystic notation.

For a ribbon tableau or supertableau $S$, let $\operatorname{Rib}(S)$ be the set of its ribbons and for $\xi \in \operatorname{Rib}(S)$, we define position of $\xi$ in the shape of $S$ as

$$
\operatorname{pos}(\xi)=\max \{j-i \mid(i, j) \in \xi\} .
$$

In [115, Proposition 3.1.2], if we take $A=[n] \cup[m]$, then we have the following result.
Theorem 6.7. Let $\lambda$ and $\mu$ are partitions of length at most tn such that $\operatorname{core}_{t}(\lambda) / \operatorname{core}_{t}(\mu)$ is empty and $n_{i}(\lambda, t n)=n_{i}(\mu, t n)=n_{i}$ for all $i \in[0, t-1]$. Then there is a bijection between the set of standard $t$-ribbon supertableaux $S$ of shape $\lambda / \mu$ with entries in $[n] \cup[m]$, and the set of t-tuples $\left(S_{0}, S_{1}, \ldots, S_{t-1}\right)$ of standard supertableaux, where $S_{i}$ has shape $\lambda^{(i)} / \mu^{(i)}$, and the sets of entries of the $S_{i}$ are mutually disjoint. If $x$ is a square of $\lambda^{(i)} / \mu^{(i)}$, then $S_{i}(x)=S(\xi)$ for a $\xi \in \operatorname{Rib}(S)$ with $\operatorname{pos}(\xi)=t\left(\operatorname{pos}(x)+n_{i}-n\right)+i$.

For a bijection similar to the above theorem involving semistandard supertableaux, without the condition of disjointness on entries, we define a standardisation of semistandard supertableau. The standardisation of a semistandard supertableau is a standard supertableau obtained from it by renumbering its entries such that the relative order of distinct entries is preserved, and equal unprimed and primed entries are made increasing from left to right and top to bottom respectively. It is well defined since the ribbons with same entries have distinct positions and ordering them by increasing position (for unprimed entries) and decreasing position (for primed entries) gives a valid standard supertableau. See Figure 6.1 for an example of standardisation. Therefore, we have the following generalization of [115, Proposition 3.2.2]


Figure 6.1: semistandard 3-ribbon supertableau and its standardization

Theorem 6.8. There is a natural bijection between the set of semistandard $t$-ribbon supertableaux $T$ of shape $\lambda / \mu$, and the set of $t$-tuples $\left(T_{0}, T_{1}, \ldots, T_{t-1}\right)$ of semistandard supertableaux, with $T_{i}$ of shape $\lambda^{(i)} / \mu^{(i)}$, and $\prod_{i=0}^{t-1} \mathrm{wt}\left(T_{i}\right)=\mathrm{wt}(T)$.

Proof. It is sufficient to show that the $T_{i}$ are semistandard supertableaux, and that the map is invertible. Let $x, y \in \lambda^{(i)} / \mu^{(i)}$ such that $T_{i}(x)=T_{i}(y)$ and $S_{i}(x)<S_{i}(y)$, and are unprimed. Suppose $\xi, \xi^{\prime} \in \operatorname{Rib}(S)=\operatorname{Rib}(T)$ by $S(\xi)=S_{i}(x)$ and $S\left(\xi^{\prime}\right)=S_{i}(y)$. Then $T(\xi)=T\left(\xi^{\prime}\right)$, so that $\operatorname{pos}(\xi)<\operatorname{pos}\left(\xi^{\prime}\right)$, while $\operatorname{pos}(\xi) \equiv \operatorname{pos}\left(\xi^{\prime}\right) \equiv i(\bmod t)$; therefore by Theorem 6.7, we have $\operatorname{pos}(x)<\operatorname{pos}(y)$. Similar argument holds if $T_{i}(x)$ and $T_{i}(y)$ are primed. So, $T_{i}$ is semistandard. For invertibility we need to order all the occurrences of the same entry in any of the tableaux $T_{i}$, in order to determine the $S_{i}$; makes clear that these occurrences $T_{i}(x)$ should be ordered by increasing value of $t\left(\operatorname{pos}(x)+n_{i}-n\right)+i$ if $T_{i}(x)$ are unprimed, and by decreasing value of $t\left(\operatorname{pos}(x)+n_{i}-n\right)+i$ if $T_{i}(x)$ are primed.

As an example, the semistandard 3-ribbon tableau and its standardisation displayed in Figure 6.1 corresponds to


Corollary 6.9. Let $\lambda$ and $\mu$ be partitions of length at most tn. Then for all divisors $d \mid t$,

$$
\prod_{i=0}^{\frac{t}{d}-1} \mathrm{hs}_{\lambda^{(i)} / \mu^{(i)}}\left(X^{t} / Y^{t}\right)=\sum_{R} \mathrm{wt}(R)
$$

where $R$ runs over the set of $\frac{t}{d}$-ribbon supertableaux of shape $\lambda / \mu$ filled with entries $[d n] \cup[d m]$.

Remark 6.10. Let $\lambda$ and $\mu$ be partitions of length at most $t n$ such that core $_{t}(\lambda) / \operatorname{core}_{t}(\mu)$ is empty. If $\operatorname{sgn}\left(\sigma_{\lambda}\right)=\operatorname{sgn}\left(\sigma_{\mu}\right)$, then by Corollary 6.6 and Corollary 6.9 at $X=$ $(1, \ldots, 1), Y=\varnothing$, for all divisors $d \mid t, s_{\lambda / \mu}\left(1, \omega^{d}, \ldots, \omega^{d(t n-1)}\right)$ is the number of $\frac{t}{d}$ ribbon tableaux of shape $\lambda / \mu$ filled with entries [dn]. Moreover, if $t$ is odd, then $\mathrm{hs}_{\lambda / \mu}\left(1, \omega^{d}, \ldots, \omega^{d(t n-1)} / 1, \omega^{d}, \ldots, \omega^{d(t m-1)}\right)$ is the number of $\frac{t}{d}$-ribbon supertableaux of shape $\lambda / \mu$ filled with entries $[d n] \cup[d m]$, for all $d \mid t$.

Corollary 6.11. Suppose $t$ is prime. Let $\phi_{\lambda / \mu}(r, s)$ be the number of $r$-ribbon tableaux of shape $\lambda / \mu$ filled with entries in $[s]$. Then $\phi_{\lambda / \mu}(1, t n)-\phi_{\lambda / \mu}(t, n)$ is a multiple of $t$.

Proof. If $\operatorname{core}_{t}(\lambda) \neq \operatorname{core}_{t}(\mu)$, then by Definition 2.12, $\phi_{\lambda / \mu}(t, n)=0$. By (2.3.8), we have

$$
s_{\lambda / \mu}\left(1, \omega, \ldots, \omega^{t n-1}\right)=\sum_{T \in \operatorname{SSYT}_{t n}(\lambda / \mu)} \omega^{\sum_{i=0}^{n-1}\left(n_{2+i t}(T)+2 n_{3+i t}(T)+\cdots+(t-1) n_{(i+1) t}(T)\right)}=0
$$

where the second equality uses Corollary 6.6 at $X=(1, \ldots, 1)$. Since $\sum_{i=0}^{t-1} a_{i} \omega^{i}=0$ for some $a_{i} \in \mathbb{Z}$ implies $a_{i}=c$ for all $i, \phi_{\lambda / \mu}(1, t n)$ is a multiple of $t$ and the corollary holds. If $\operatorname{core}_{t}(\lambda)=\operatorname{core}_{t}(\mu)$, then by Remark 6.10.

$$
\begin{aligned}
\phi_{\lambda / \mu}(1, t n) & -\phi_{\lambda / \mu}(t, n) \\
& =\sum_{T \in \operatorname{SSYT}_{t n}(\lambda / \mu)}\left(1-\operatorname{sgn}\left(\sigma_{\lambda}\right) \operatorname{sgn}\left(\sigma_{\mu}\right) \omega^{\sum_{i=0}^{n-1}\left(n_{2+i t}(T)+2 n_{3+i t}(T)+\cdots+(t-1) n_{(i+1) t}(T)\right)}\right) .
\end{aligned}
$$

Since $\phi_{\lambda / \mu}(1, t n)-\phi_{\lambda / \mu}(t, n)$ is an integer, and $\sum_{i=0}^{t-1} a_{i} \omega^{i}=0$ for some $a_{i} \in \mathbb{Z}$ implies $a_{i}=c$ for all $i, \phi_{\lambda / \mu}(1, t n)-\phi_{\lambda / \mu}(t, n)$ is a multiple of $t$. This completes the proof.

Remark 6.12. A result similar to Corollary 6.11 holds for the number of supertableaux. Suppose $t$ is an odd prime. Let $\psi_{\lambda / \mu}(r, s / u)$ be the number of $r$-ribbon supertableaux of shape $\lambda / \mu$ filled with entries in $[s] \cup[u]$. Then $\psi_{\lambda / \mu}(1, t n / t m)-\psi_{\lambda / \mu}(t, n / m)$ is a multiple of $t$.

### 6.2 Combinatorial proof of skew Schur factorization at $t=2$

We now give a combinatorial proof of the Corollary 6.6 when $t=2$. We recall the definition of domino tableau and coverable tableau from Remark 2.13 and Definition 2.15 respectively. For a domino tableau $D \in \mathcal{D}_{n}(\lambda / \mu)$, let $X^{D}=x_{1}^{2 d_{1}} x_{2}^{2 d_{2}} \ldots x_{n}^{2 d_{n}}$, here $d_{i}$ is the number of dominoes filled with the entry $i$.

Theorem 6.13 ([115, Corollary 3.2.3]). Let $\lambda$ and $\mu$ be partitions of length at most $2 n$. Then

$$
s_{\lambda(0) / \mu^{(0)}}\left(X^{2}\right) s_{\lambda^{(1)} / \mu^{(1)}}\left(X^{2}\right)=\sum_{D} X^{D},
$$

where $D$ runs over the set of domino tableaux of shape $\lambda / \mu$.

Proof of Corollary 6.6. The required skew Schur polynomial is given by

$$
s_{\lambda / \mu}(X,-X)=\sum_{T}(X,-X)^{T},
$$

where the sum is over all semi-standard Young tableau $T$ of shape $\lambda / \mu$ filled with entries in $\{1,2, \ldots, 2 n\}$. Suppose $\mathcal{C}_{2 n}(\lambda / \mu)$ be the set of coverable tableaux of shape $\lambda / \mu$ filled
with entries in $\{1,2, \ldots, 2 n\}$. Then

$$
\begin{equation*}
s_{\lambda / \mu}(X,-X)=\sum_{T \in \mathcal{C}_{2 n}(\lambda / \mu)}(X,-X)^{T}+\sum_{T \notin \mathcal{C}_{2 n}(\lambda / \mu)}(X,-X)^{T} . \tag{6.2.1}
\end{equation*}
$$

By Lemma 2.18, we have

$$
\sum_{T \notin \mathcal{C}_{2 n}(\lambda / \mu)}(X,-X)^{T}=0 .
$$

If $\operatorname{core}_{2}(\lambda / \mu)$ is non-empty, then $\mathcal{C}_{2 n}(\lambda / \mu)=\varnothing$. So, by (6.2.1), $s_{\lambda / \mu}(X,-X)=0$. Otherwise, by Lemma 2.16, we have

$$
\sum_{T \in \mathcal{C}_{2 n}(\lambda / \mu)}(X,-X)^{T}=\sum_{D \in \mathcal{\mathcal { D } _ { n }}(\lambda / \mu)}(-1)^{\mathrm{ht}(D)} X^{D}=\operatorname{sgn}\left(\sigma_{\lambda}\right) \operatorname{sgn}\left(\sigma_{\mu}\right) \sum_{D} X^{D},
$$

where the last equality comes from Lemma 2.14 for $t=2$ and Lemma 6.3. Then using Theorem 6.13 completes the proof.

### 6.3 Cyclic sieving phenomenon

Let $C_{t}$ be the cyclic group of order $t$ acting on a finite set $X$ and $f(q)$ a polynomial with nonnegative integer coefficients. Then the triple $\left(X, C_{t}, f(q)\right)$ is said to exhibit the cyclic sieving phenomenon (CSP) if, for any integer $k \geqslant 0$,

$$
\begin{equation*}
\left|\left\{x \in X \mid \sigma^{k} \cdot x=x\right\}\right|=f\left(\omega^{k}\right) \tag{6.3.1}
\end{equation*}
$$

where $\sigma$ is a generator of $C_{n}$ and $\omega$ is a primitive $t^{\text {th }}$ root of unity.

Theorem 6.14. [95, Theorem 11.1] The triple

$$
\left(\mathrm{SSYT}_{t m}((k)), C_{t}, h_{k}\left(1, q, \ldots, q^{t m-1}\right)\right)
$$

exhibits the cyclic sieving phenomenon. If, in addition, $t$ is odd then the triple

$$
\left(\binom{[t n]}{k}, C_{t}, e_{k}\left(1, q, \ldots, q^{t n-1}\right)\right)
$$

where $\binom{[t n]}{k}$ is the set of $k$-element subsets of $[t n]$, exhibits the cyclic sieving phenomenon.

By [73, Remark 3.3] and (2.5.2), we have the following corollary.

Corollary 6.15. If $t$ is odd, then

$$
\left(\mathrm{SSYT}_{t n / t m}((k)), C_{t}, H_{k}\left(1, q, \ldots, q^{t n-1} / 1, q, \ldots, q^{t m-1}\right)\right)
$$

exhibits the cyclic sieving phenomenon.

Lemma 6.16. [3, Theorem 2.7] Suppose $f(q) \in \mathbb{Z}_{\geqslant 0}[q]$ and $f\left(\omega^{j}\right) \in \mathbb{Z}_{\geqslant 0}$, for each $j \in\{1, \ldots, t\}$. Let $X$ be any set of size $f(1)$. Then there exists an action of the cyclic group $C_{t}$ of order $t$ on $X$ such that $\left(X, C_{t}, f(q)\right)$ exhibits the cyclic sieving phenomenon if and only if for each $d \mid t$,

$$
\begin{equation*}
f\left(\omega^{d}\right)=\sum_{j \mid d} j c_{j} \tag{6.3.2}
\end{equation*}
$$

for some nonnegative integers $c_{j}$.
Recall the definition of $\sigma_{\lambda}$ from (6.1.3).
Theorem 6.17. Let $\lambda$ and $\mu$ be partitions of length at most tn such that $\operatorname{sgn}\left(\sigma_{\lambda}\right)=$ $\operatorname{sgn}\left(\sigma_{\mu}\right)$. Then there exists an action of the cyclic group $C_{t}$ of order $t$ such that the triple

$$
\begin{equation*}
\left(\operatorname{SSYT}_{t n}(\lambda / \mu), C_{t}, s_{\lambda / \mu}\left(1, q, \ldots, q^{t n-1}\right)\right) \tag{6.3.3}
\end{equation*}
$$

exhibits the cyclic sieving phenomenon.

Proof. Let $f(q)=s_{\lambda / \mu}\left(1, q, \ldots, q^{\text {tn-1 }}\right)$. Since $f(q)$ is given by (2.3.8), $f(q) \in \mathbb{Z}_{\geqslant 0}[q]$. By Lemma 6.16, it is sufficient to show for each $d \mid t$ there exists $c_{d} \geqslant 0$ such that 6.3.2) holds.

We prove this by induction on $t$. If $t=2$, then take $c_{1}=\phi_{\lambda / \mu}(2, a)$ and $2 c_{2}=$ $\phi_{\lambda / \mu}(1,2 a)-\phi_{\lambda / \mu}(2, a) \geqslant 0$, as derived in Corollary 6.11 at $t=2$. Assume that the result holds for all positive integers less than $t$. Fix $t$. If $\lambda / \mu=(k)$, then by Theorem 6.14 and Lemma 6.16, for all $d \mid t$,

$$
h_{k}\left(1, \omega^{d}, \ldots, \omega^{d(t n-1)}\right)=\sum_{j \mid d} j a_{j}^{k},
$$

for some non negative integers $a_{j}^{k}$. Since $\left(\sum_{j \mid d} j p_{j}\right)\left(\sum_{j \mid d} j q_{j}\right)=\sum_{j \mid d} j r_{j}$, where $r_{j}=$ $\sum_{i \mid d, i<j} i\left(p_{i} q_{j}+p_{j} q_{i}\right)$, by the Jacobi-Trudi identity $(2.3 .9)$, for all $d \mid t$, we see that

$$
\begin{equation*}
f\left(\omega^{d}\right)=\operatorname{det}\left(\sum_{j \mid d} j a_{j}^{\lambda_{i}-\mu_{j}-i+j}\right)=\sum_{j \mid d} j c_{j} . \tag{6.3.4}
\end{equation*}
$$

Therefore,

$$
f(q) \equiv \sum_{j \mid t} c_{j}\left(1+q^{\frac{t}{j}}+\cdots+q^{\frac{t}{j}(j-1)}\right) \quad\left(\bmod q^{t}-1\right)
$$

Since $f(q) \in \mathbb{Z}_{\geqslant 0}[q], c_{t} \geqslant 0$. Fix $d<t$. Then by Corollary 6.6 at $X=(\underbrace{1, \ldots, 1}_{d n})$,

$$
f\left(\omega^{d}\right)= \begin{cases}0 & \operatorname{core}_{\frac{t}{d}}(\lambda) \neq \operatorname{core}_{\frac{t}{d}}(\mu),  \tag{6.3.5}\\ \prod_{i=0}^{\frac{t}{d}-1} s_{\lambda^{(i)} / \mu^{(i)}}(\underbrace{1, \ldots, 1}_{d n}) & \operatorname{core}_{\frac{t}{d}}(\lambda)=\operatorname{core}_{\frac{t}{d}}(\mu) .\end{cases}
$$

Since $d<t$, by inductive argument, for all $e \mid d$ and $i \in\left[0, \frac{t}{d}-1\right]$,

$$
\begin{equation*}
s_{\lambda^{(i)} / \mu^{(i)}}\left(1, \omega^{t e / d}, \ldots, \omega^{t e(d n-1) / d}\right)=\sum_{j \mid e} j d_{j}^{(i)}, \tag{6.3.6}
\end{equation*}
$$

for some nonnegative integers $d_{j}^{(i)}$. If $\operatorname{core}_{\frac{t}{d}}(\lambda)=\operatorname{core}_{\frac{t}{d}}(\mu)$, then take $e=d$ in 6.3.6 and substitute in 6.3.5) to get

$$
f\left(\omega^{d}\right)=\prod_{i=0}^{\frac{t}{d}-1}\left(\sum_{j \mid d} j d_{j}^{(i)}\right)=\sum_{j \mid d} j c_{j},
$$

where the last equality uses (6.3.4). The uniqueness of $c_{j}$ implies $c_{j} \geqslant 0$ for all $j \mid d$. This completes the proof.

We now state our final result and give a sketch of the proof following similar ideas as in the proof of Theorem 6.17.

Theorem 6.18. Suppose $t$ is odd. Let $\lambda$ and $\mu$ be partitions of length at most tn such that $\operatorname{sgn}\left(\sigma_{\lambda}\right)=\operatorname{sgn}\left(\sigma_{\mu}\right)$. Then there exists an action of the cyclic group $C_{t}$ of order $t$ such that the triple

$$
\left(\mathrm{SSYT}_{t n / t m}(\lambda / \mu), C_{t}, \mathrm{hs}_{\lambda / \mu}\left(1, q, \ldots, q^{t n-1} / 1, q, \ldots, q^{t m-1}\right)\right)
$$

exhibits the cyclic sieving phenomenon.
Proof. Let $f(q)=\mathrm{hs}_{\lambda / \mu}\left(1, q, \ldots, q^{t n-1} / 1, q, \ldots, q^{t m-1}\right)$. Since $f(q)$ is given by (2.5.4), $f(q) \in \mathbb{Z}_{\geqslant 0}[q]$. We apply Lemma 6.16 to prove the result.

The proof proceeds by induction on $t$. If $t=3$, then take $c_{1}=\psi_{\lambda / \mu}(3, a)$ and $3 c_{3}=\psi_{\lambda / \mu}(1,3 a)-\psi_{\lambda / \mu}(3, a) \geqslant 0$, as in Remark 6.12 at $t=3$. Assume that the result
holds for all odd integers less than $t$. Fix $t$. If $\lambda / \mu=(k)$, then by Corollary 6.15 and Lemma 6.16, for all $d \mid t$,

$$
H_{k}\left(1, \omega^{d}, \ldots, \omega^{d(t n-1)} / 1, \omega^{d}, \ldots, \omega^{d(t m-1)}\right)=\sum_{j \mid d} j a_{j}^{k},
$$

for some non-negative integers $a_{j}^{k}$. Then by the Jacobi-Trudi identity (2.5.6), for all $d \mid t$, we see that

$$
\begin{equation*}
f\left(\omega^{d}\right)=\operatorname{det}\left(\sum_{j \mid d} j a_{j}^{\lambda_{i}-\mu_{j}-i+j}\right)=\sum_{j \mid d} j c_{j} . \tag{6.3.7}
\end{equation*}
$$

Therefore,

$$
f(q) \equiv \sum_{j \mid t} c_{j}\left(1+q^{\frac{t}{j}}+\cdots+q^{\frac{t}{j}(j-1)}\right) \quad\left(\bmod q^{t}-1\right)
$$

Since $f(q) \in \mathbb{Z}_{\geqslant 0}[q], c_{t} \geqslant 0$. Fix $d<t$. Then by Theorem 6.5, at $X=(\underbrace{1, \ldots, 1}_{d n})$ and $Y=(\underbrace{1, \ldots, 1}_{d m})$, we have

$$
f\left(\omega^{d}\right)= \begin{cases}0 & \operatorname{core}_{\frac{t}{d}}(\lambda) \neq \operatorname{core}_{\frac{t}{d}}(\mu),  \tag{6.3.8}\\ \prod_{i=0}^{\frac{t}{d}-1} \mathrm{hs}_{\lambda^{(i)} / \mu^{(i)}}(\underbrace{1, \ldots, 1}_{d n} / \underbrace{1, \ldots, 1}_{d m}) & \operatorname{core}_{\frac{t}{d}}(\lambda)=\operatorname{core}_{\frac{t}{d}}(\mu) .\end{cases}
$$

Since $d<t$ and $d$ is odd, by inductive argument, for all $e \mid d$ and $i \in\left[0, \frac{t}{d}-1\right]$,

$$
\begin{equation*}
\mathrm{hs}_{\lambda^{(i)} / \mu^{(i)}}\left(1, \omega^{t e / d}, \ldots, \omega^{t e(d n-1) / d} / 1, \omega^{t e / d}, \ldots, \omega^{t e(d m-1) / d}\right)=\sum_{j \mid e} j d_{j}^{(i)} \tag{6.3.9}
\end{equation*}
$$

for some nonnegative integers $d_{j}^{(i)}$. If $\operatorname{core}_{\frac{t}{d}}(\lambda)=\operatorname{core}_{\frac{t}{d}}(\mu)$, then take $e=d$ in (6.3.9) and substitute in (6.3.8) to get

$$
f\left(\omega^{d}\right)=\prod_{i=0}^{\frac{t}{d}-1}\left(\sum_{j \mid d} j d_{j}^{(i)}\right)=\sum_{j \mid d} j c_{j}
$$

where the last equality uses (6.3.7). The uniqueness of $c_{j}$ implies $c_{j} \geqslant 0$ for all $j \mid d$. This completes the proof.

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[^0]:    ${ }^{1}$ While this terminology also seems to be have been used for partitions whose odd parts have even multiplicity [54, it does not seem widespread.

