On the Chern number of good filtrations of ideals

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Virtual Symposium in honour of Dilip Patil

29 July 2021

Outline

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6. Huckaba-Marley Theorem for $e_1(\mathcal{F})$ for an I-good filtration $\mathcal{F}$ in a Cohen-Macaulay local ring.
7. On a question of C. Huneke about F-rational rings.

Examples of I-good filtrations

1. **Definition.** Suppose I is an ideal of a Noetherian ring R.
2. A sequence of ideals $\mathcal{F} = \{I_n\}_{n\in\mathbb{Z}}$ is called an I-**filtration** if for all $m, n \in \mathbb{Z}$, (i) $I_{n+1} \subseteq I_n$, (ii) $I_m I_n \subseteq I_{m+n}$, (iii) $I^n \subseteq I_n$.
3. An I-filtration is called I-**good** if $\exists\, k$ so that $I_{n+k} \subseteq I^n\ \forall n \in \mathbb{Z}$.

1. **Examples.** The I-adic filtration $\{I^n\}$ is I-good.
2. We say that $x \in R$ is **integral** over I if

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0 \text{ for some } a_i \in I^i \text{ for all } i.$$

3. The integral closure of I is $\overline{I} = \{x \in R \mid x \text{ is integral over } I\}$.
4. **Theorem. [D. Rees, 1961]** Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then R is analytically unramified
   $\iff$ the filtration $\{\overline{I^n}\}$ is I-good for all ideals I,
   $\iff$ there exists and $\mathfrak{m}$-primary ideal I so that $\{\overline{I^n}\}$ is I-good.

Hilbert function and polynomial of an I-good filtration

1. **Definition.** Let R be a Noetherian ring of prime characteristic p and $q = p^e$. Let $\min(R) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r\}$ be the set of minimal primes of R and $R^\circ = R \setminus \cup_{i=1}^r \mathfrak{p}_i$. Let $I = (a_1, a_2, \ldots, a_n)$.
2. The q^{th} **Frobenius power** of I is the ideal $I^{[q]} = (a_1^q, a_2^q, \ldots, a_n^q)$.
3. The **tight closure** I^* of an ideal I is the ideal

$$I^* = \{x \in R \mid \text{ there exists } c \in R^\circ \text{ so that } cx^q \in I^{[q]} \text{ for all large } q\}.$$

4. **Definition.** An element $c \in R^\circ$ is called a **test element** if whenever $x \in I^*$ then $cx^q \in I^{[q]}$ for all q and all ideal I of R.
5. Since $I \subseteq I^* \subseteq \overline{I}$, if R is analytically unramified, $\{(I^n)^*\}$ is I-good.
6. **Definition.** Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. Let I be an $\mathfrak{m}$-primary ideal of R. The **Hilbert function** of an I-good filtration $\mathcal{F} = \{I_n\}$ is defined as: $H_{\mathcal{F}}(n) = \lambda(R/I_n)$.
7. **Theorem.(Rees)** There exists a polynomial $P_{\mathcal{F}}(x) \in \mathbb{Q}[x]$ called the **Hilbert polynomial** of $\mathcal{F}$ so that $H_{\mathcal{F}}(n) = P_{\mathcal{F}}(n)$ for all large n.

The normal and the tight Hilbert polynomial of an ideal

1. **Definitions.** The Hilbert polynomial of $\mathcal{F}$ is written as

$$P_{\mathcal{F}}(x) = e_0(\mathcal{F})\binom{x+d-1}{d} - e_1(\mathcal{F})\binom{x+d-2}{d-1} + \cdots + (-1)^d e_d(\mathcal{F}).$$

2. $e_i(\mathcal{F})$ are called the **Hilbert coefficients** of $\mathcal{F} = \{I_n\}$.
3. If $\mathcal{F} = \{I^n\}$ then we write $P_{\mathcal{F}}(x) = P_I(x), \;\; e_i(\mathcal{F}) = e_i(I)$.
4. $P_I(x)$ is called the **Hilbert polynomial** of I.
5. If $\mathcal{F} = \{\overline{I^n}\}$ then we write $P_{\mathcal{F}}(x) = \overline{P}_I(x), \;\; e_i(\mathcal{F}) = \overline{e}_i(I)$.
6. $\overline{P}_I(x)$ is called the **normal Hilbert polynomial** of I.
7. If $\mathcal{F} = \{(I^n)^*\}$ then we write $P_{\mathcal{F}}(x) = P_I^*(x), \;\; e_i(\mathcal{F}) = e_i^*(I)$.
8. $P_I^*(x)$ is called the **tight Hilbert polynomial** of I.
9. **Definition.** The coefficient $e_1(\mathcal{F})$ is called the **Chern number** of $\mathcal{F}$.

Graded modules and algebras for I-filtrations

1. **Definition.** Let $\mathcal{F} = \{I_n \mid n \in \mathbb{Z}\}$ be an I-filtration. By convention $I_n = R$ for all $n \leq 0$. Let t be indeterminate.

$$
\begin{array}{rcccl}
\textbf{Rees algebra of } \mathcal{F} & = & \mathcal{R}(\mathcal{F}) & = & \bigoplus_{n=0}^{\infty} I_n t^n \\
\text{mm } \textbf{Extended Rees algebra of } \mathcal{F} & = & \mathcal{R}'(\mathcal{F}) & = & \bigoplus_{n\in\mathbb{Z}} I_n t^n \\
\text{mm } \textbf{Associated graded ring of } \mathcal{F} & = & G(\mathcal{F}) & = & \bigoplus_{n=0}^{\infty} I_n/I_{n+1}
\end{array}
$$

2. If $\mathcal{F} = \{I^n\}$ then these algebras are denoted by $\mathcal{R}(I), \mathcal{R}'(I)$, and $G(I)$.
3. **Theorem.** Let $(R, \mathfrak{m})$ be a d-dimensional local ring and $\mathcal{F} = \{I_n\}$ be an I-good filtration for an $\mathfrak{m}$-primary ideal I. Then $G(\mathcal{F})$ is a finitely generated $G(I)$-module. Moreover,

$$\dim \mathcal{R}'(\mathcal{F}) = d + 1, \ \dim G(\mathcal{F}) = d \text{ and } \dim \mathcal{R}(\mathcal{F}) = d + 1.$$

4. An ideal $J \subset I_1$ is called a **reduction** of $\mathcal{F}$ if $JI_n = I_{n+1}$ for all large n.

Results about $e_1(I)$

1. **Theorem.** **(Northcott. 1960)** Let R be a Cohen-Macaulay local ring and I be an $\mathfrak{m}$-primary ideal. Then $e_1(I) \geq 0$ with equality $\iff$ I is generated by a regular sequence.

Conjectures of W. Vasconcelos, 2008

2. **The negativity conjecture.** For any ideal Q generated by a system of parameters, $e_1(Q) < 0$ if and only if R is not Cohen-Macaulay.
3. **Theorem.** **(Mandal-Singh-Verma, 2010)** Let R be a d-dimensional Noetherian local ring. Let J be an ideal generated by a system of parameters. Then $e_1(J) \leq 0$.
4. Partial solutions for the negativity conjecture were given by L. Ghezzi, J. Hong and W. Vasconcelos in 2009 and by M. Mandal, B. Singh and J. Verma in 2011.
5. **Definition.** A Noetherian local ring is called **formally unmixed** if for any associated prime $\mathfrak{p}$ of the $\mathfrak{m}$-adic completion $\hat{R}$ $\dim \hat{R}/\mathfrak{p} = \dim R$.
6. L Ghezzi, S. Goto, J. Hong, T. T. Phuong, W. V. Vaconcelos settled the negativity conjecture in 2010 by proving the following result.
7. **Theorem.** A formally unmixed local ring is Cohen-Macaulay if and only if $e_1(Q) = 0$ for some parameter ideal Q.

Bounds for the Chern number of the $\{\overline{(I^n)}\}$ filtration

1. **Theorem. [Huckaba-Marley, 1997]** Let $(R, \mathfrak{m})$ be a d-dimensional CM local ring. Let I be an $\mathfrak{m}$-primary ideal and $\mathcal{F}$ be an I-good filtration. Let J be a minimal reduction of $\mathcal{F}$.
   (1) $e_1(\mathcal{F}) \geq \sum_{n\geq 1} \lambda(I_n/(J \cap I_n))$, with equality iff $G(\mathcal{F})$ is CM.
   (2) $e_1(\mathcal{F}) \leq \sum_{n\geq 1} \lambda(I_n/JI_{n-1})$ with equality iff depth $G(\mathcal{F}) \geq d-1$.
2. **Corollary.** $e_1(\mathcal{F}) = 0 \iff I_n = J^n$ for all n.
3. **Corollary.** Let R be a Cohen-Macaulay analytically unramified local ring and I be an $\mathfrak{m}$-primary ideal. If $\overline{e}_1(I) = 0$ then R is a regular local ring, I is generated by a regular sequence and it is a normal ideal.
4. **The positivity conjecture of Vasconcelos.** For any $\mathfrak{m}$-primary ideal I, of an analytically unramified local ring, $\overline{e}_1(I) \geq 0$.
5. **Theorem. (Mandal-Singh-Verma, 2011)** The positivity conjecture is true for (1) 2-dimensional complete local domains (2) for analytically unramified local ring R so that there is a Cohen-Macaulay local ring S containing R and S/R has finite length and (3) the integral closure of R is a finite Cohen-Macaulay R-module.
6. **Theorem. (Mandal-Hong-Goto, 2011)** The positivity conjecture is true for formally unmixed analytically unramified local rings.

F-rational local rings

1. **Definition.** A Noetherian ring R of prime characteristic is called **weakly** F-**regular** if all ideals of R are tightly closed. If $R_{\mathfrak{p}}$ is weakly F-regular for all prime ideals $\mathfrak{p}$ of R then R is called F-regular.
2. **Examples.** Regular local rings, polynomial rings over a field, direct summands of F-regular rings, are all F-regular.
3. **Definition.** An ideal I of a Noetherian ring is called a parameter ideal if I can be generated by ht I elements. A Noetherian ring R is called F-**rational** if all parameter ideals are tightly closed. If R is a homomorphic image of a CM ring and it is F-rational then it is normal and CM and its $\mathfrak{m}$-adic completion and localizations are F-rational.
4. **Examples.** Let k be a field of prime characteristic $p, S = k[X, Y, Z]$.
(1) $S/(X^2 + Y^2 + Z^2)$ is F-rational if $p \geq 3$.
(2) $S/(X^2 - Y^3 - Z^7)$ is not F-rational.
(3) $S/(X^2 - Y^3 - Z^5)$ is F-rational iff $p \geq 11$.
(4) If $p \geq 11, 1/a + 1/b + 1/c > 1$ then $S/(X^a + Y^b + Z^c)$ is F-rational.

Vanishing of $e_1^*(Q)$ and F-rational local rings

1. **Theorem. (K. Goel, V. Mukundan and J. K. Verma, 2020)** Let R be a Cohen-Macaulay analytically unramified local ring of prime characteristic and I be generated by a system of parameters of R. Then $e_1^*(I) = 0 \iff R$ is an F-rational local ring.
2. **Question. (C. Huneke)** Let $(R, \mathfrak{m})$ be a formally unmixed local Noetherian ring and Q be an ideal generated by a system of parameters. Is it true that $e_1^*(Q) = 0 \iff R$ is F-rational?
3. **Answer. (S. Dubey, P. H. Quy and J. K. Verma, 2021)** We construct a complete local domain of dimension 2 that is not F-rational but there is an $\mathfrak{m}$-primary parameter ideal Q and $e_1^*(Q) = 0$.
4. **Example.** Let k be a field, char $k = p \geq 3$ and $R = k[[x^4, x^3y, xy^3, y^4]]$. Then $\overline{R} = S = k[[x^4, x^3y, x^2y^2, xy^3, y^4]]$ is Cohen-Macaulay and F-regular.
5. We have $C := S/R \cong k$, so that $\ell(C/JC) = 1$ for any $\mathfrak{m}$-primary ideal J of R. Let Q be any $\mathfrak{m}$-primary parameter ideal of R.

A characterization of F-rational local rings

1. Consider the short exact sequence,
$$0 \to R/(Q^{n+1})^* \to S/(Q^{n+1}S)^* \to C \to 0.$$
Then $\ell(R/(Q^{n+1})^*) = \ell(S/(Q^{n+1})^*S) - 1.$
2. Since S is F-regular, $\ell(R/(Q^{n+1})^*) = e_0(Q)\binom{n+2}{2} - 1$, for all $n \geq 1$. Since $S/\mathfrak{n} \cong R/\mathfrak{m}$, $e_0(Q) = e_0(QS)$. Hence $e_1^*(Q) = 0$. But R is not even CM.
3. **Theorem.(S. Dubey, P. H. Quy and J. K. Verma, 2021)** Let $(R, \mathfrak{m})$ be an excellent reduced equidimensional local ring of prime characteristic p and dimension $d \geq 2$. Let $x_1, x_2, \ldots, x_d$ be test elements and $Q = (x_1, x_2, \ldots, x_d)$ be $\mathfrak{m}$-primary. Then R is F-rational. $\iff$ $e_1^*(Q) = 0$ and depth $R \geq 2$.
4. The following recent result due to Linquan Ma and Pham Hung Quy plays a crucial role for proving the above theorem.
5. **Theorem.** Let $(R, \mathfrak{m})$ be an excellent equidimensional local ring such that the test ideal $\tau_{par}(R)$ for all parameter ideals is $\mathfrak{m}$-primary. Let Q be an ideal generated by a system of parameters contained in $\tau_{par}(R)$. Then we have
$$\ell(Q^*/Q) = \sum_{i=0}^{d-1} \binom{d}{i} \ell(H^i_{\mathfrak{m}}(R)) + \ell(0^*_{H^d_{\mathfrak{m}}(R)}).$$

Sketch of a proof of the main theorem

1. If Q is an ideal generated by a system of parameters of R consisting of test elements then it is a standard system of parameters of R. This means

$$\ell(R/Q) - e(Q) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell(H^i_{\mathfrak{m}}(R)).$$

2. If Q is generated by a standard system of parameters, then for all $n \geq 0$,

$$\ell(R/Q^n) = \sum_{i=0}^{d} (-1)^i e_i(Q) \binom{n+d-1-i}{d-i}, \text{ where}$$

$$e_i(Q) = (-1)^i \sum_{j=0}^{d-i} \binom{d-i-1}{j-1} \ell(H^j_{\mathfrak{m}}(R)) \text{ for all } i = 1, 2, \ldots, d.$$

$$e_1^*(Q) = \sum_{i=1}^{d-1} \binom{d-1}{i-1} \ell(H^i_{\mathfrak{m}}(R)) + \ell(0^*_{H^d_{\mathfrak{m}}(R)}).$$

3. Now we use a characterization of F-rational rings due to Karen Smith: A Cohen-Macaulay excellent local ring is F-rational if and only if $0^*_{H^d_{\mathfrak{m}}(R)} = 0$.