Differential Methods for 0-dimensional Schemes

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Contents

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1. Zero-dimensional Schemes

You can teach an old dog new tricks
if the old dog wants to learn.
(Tip O'Neill)

$P = K[x_0, \ldots, x_n]$ polynomial ring over a field K of characteristic 0

$I = \langle f_1, \ldots, f_m \rangle$ homogeneous saturated ideal in P

$\mathbb{P}^n$ projective space over $\overline{K}$

$\mathbb{X} = \mathcal{Z}(I) \subseteq \mathbb{P}^n$ 0-dimensional subscheme

$R = P/I$ homogeneous coordinate ring of $\mathbb{X}$ is a 1-dimensional Cohen-Macaulay ring

$x_0 \in R$ is a assumed to be a non-zerodivisor

The Hilbert Function

The map $\mathrm{HF}_{\mathbb{X}} : \mathbb{Z} \longrightarrow \mathbb{Z}$ given by $\mathrm{HF}_{\mathbb{X}}(i) = \dim_K(R_i)$ is called the **Hilbert function** of $\mathbb{X}$. It satisfies

$1 = \mathrm{HF}_{\mathbb{X}}(0) < \mathrm{HF}_{\mathbb{X}}(1) < \cdots < \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}}) = \deg(\mathbb{X}) = \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}}+1) = \cdots$ where $r_{\mathbb{X}}$ is called the **regularity index** of $\mathbb{X}$

Theorem (Bigatti, Geramita)

Given a set of points $\mathbb{X}$ in $\mathbb{P}^n$, the following claims hold:

(a) *At most $r_{\mathbb{X}}+1$ points of $\mathbb{X}$ are collinear.*

(b) *If $\mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}}) = \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}}-1)+1 = \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}}-2)+2$ then precisely $r_{\mathbb{X}}+1$ points of $\mathbb{X}$ are collinear.*

The Canonical Module

The graded R-module $\omega_R = \underline{\mathrm{Hom}}_{K[x_0]}(R, K[x_0])(-1)$ is called the **canonical module** of R. We have

$$\mathrm{HF}_{\omega_R}(-r_{\mathbb{X}}) = 0 < \mathrm{HF}_{\omega_R}(-r_{\mathbb{X}}+1) < \cdots < \mathrm{HF}_{\omega_R}(1) = \deg(\mathbb{X}) = \cdots$$

Theorem (Geramita, K, Robbiano)

For a finite set of points $\mathbb{X}$ in $\mathbb{P}^n$, we have equivalent conditions:

(a) *The set $\mathbb{X}$ has the* **Cayley-Bacharach property**, *i.e., every hypersurface of degree $r_{\mathbb{X}}-1$ which passes through all points of $\mathbb{X}$ but one, automatically passes through the remaining point.*

(b) *The multiplication map $R_{r_{\mathbb{X}}-1} \otimes (\omega_R)_{-r_{\mathbb{X}}+1} \longrightarrow (\omega_R)_0$ is non-degenerate.*

2. Kähler Differentials

"So, what's your superpower?"

"I'm rich."

(Tony Stark)

$\mathbb{X} \subset \mathbb{P}^n$ 0-dimensional subscheme

$R = P/I_{\mathbb{X}} = K[x_0, \dots, x_n]/I_{\mathbb{X}}$ homogeneous coordinate ring of $\mathbb{X}$

$\mu : R \otimes_K R \longrightarrow R$ multiplication map

$J = \ker(\mu) = \langle x_i \otimes 1 - 1 \otimes x_i \mid i = 0, \dots, n \rangle$

The finitely generated graded R-module $\Omega^1_{R/K} = J/J^2$ is called the **module of Kähler differentials** of R/K (or of $\mathbb{X}$).

The map $d : R \longrightarrow \Omega^1_{R/K}$ given by $df = f \otimes 1 - 1 \otimes f + J^2$ is called the **universal derivation** of R/K.

Computing $\Omega^1_{R/K}$

For $P = K[x_0, \ldots, x_n]$, we have $\Omega^1_{P/K} = P\, dx_0 \oplus \cdots \oplus P\, dx_n$.

Theorem

We have $\Omega^1_{R/K} = \Omega^1_{P/K}/(I_{\mathbb{X}}\Omega^1_{P/K} + dI_{\mathbb{X}})$. In other words, there is a homogeneous exact sequence

$$0 \longrightarrow \mathcal{G}(-1) \longrightarrow R^{n+1}(-1) \longrightarrow \Omega^1_{R/K} \longrightarrow 0$$

where $\mathcal{G}$ is generated by the tuples $(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n})$ with $f \in I_{\mathbb{X}}$ and where $(g_0, \ldots, g_n)$ is mapped to $g_0 dx_0 + \cdots + g_n dx_n$ on the right-hand side.

The Hilbert Function of $\Omega^1_{R/K}$

For $i \in \mathbb{Z}$, let $\mathrm{HF}_{\Omega^1_{R/K}}(i) = \dim_K(\Omega^1_{R/K})_i$. The map $\mathrm{HF}_{\Omega^1_{R/K}} : \mathbb{Z} \to \mathbb{Z}$ is called the **Hilbert function** of $\Omega^1_{R/K}$.

Theorem

(a) $\mathrm{HF}_{\Omega^1_{R/K}}(i) = 0$ *for* $i \leq 0$.

(b) $\mathrm{HF}_{\Omega^1_{R/K}}(i)$ *has a constant value* $\mathrm{HP}_{\Omega^1} := \mathrm{HP}(\Omega^1_{R/K})$ *for* $i \gg 0$.

(c) *Let* $\mathrm{ri}_{\Omega^1} := \mathrm{ri}(\Omega^1_{R/K})$ *be the* **regularity index** *of* $\Omega^1_{R/K}$*, i.e., the smallest number* j *such that* $\mathrm{HF}_{\Omega^1_{R/K}}(i) = \mathrm{HP}(\Omega^1_{R/K})$ *for* $i \geq j$*. Then we have* $\mathrm{ri}(\Omega^1_{R/K}) \geq r_{\mathbb{X}} + 1$ *and if* $\mathrm{ri}(\Omega^1_{R/K}) > r_{\mathbb{X}} + 1$ *then*

$$\mathrm{HF}_{\Omega^1_{R/K}}(r_{\mathbb{X}} + 1) > \cdots > \mathrm{HF}_{\Omega^1_{R/K}}(\mathrm{ri}_{\Omega^1})$$

Kähler Differential m-Forms

For $m \geq 0$, we let $\Omega^m_{R/K} = \Lambda^m_R \ \Omega^1_{R/K}$ and call it the **module of Kähler differential m-forms** of R/K (or of $\mathbb{X}$).

The exterior algebra $\Lambda_R \, \Omega^1_{R/K} = \bigoplus_{i \geq 0} \Omega^m_{R/K}$ is called the **Kähler differential algebra** of R/K (or of $\mathbb{X}$).

Theorem (Computing $\Omega^m_{R/K}$)

For every $m \geq 1$, we have $\Omega^m_{R/K} = \Omega^m_{P/K}/(I_{\mathbb{X}}\Omega^m_{P/K} + dI_{\mathbb{X}} \wedge \Omega^{m-1}_{P/K})$.

This allows us to compute a presentation of $\Omega^m_{R/K}$. The finitely generated graded R-module $\Omega^m_{R/K}$ has a constant Hilbert polynomial HP_{Ω^m} and a regularity index ri_{Ω^m} which we can compute as well.

Example

In the projective plane $\mathbb{P}^2$ over $K = \mathbb{Q}$, let $\mathbb{X}$ be a set of 6 points on an irreducible conic, and let $\mathbb{Y}$ be a set of 6 points on a reducible conic, e.g., $\mathbb{Y} = \{(1:-1:0),(1:1:0),(1:2,0),(1:0:-1),(1:0:1),(1:0:2)\} \subset \mathcal{Z}(x_1x_2)$. Then we have $\mathrm{HF}_{\mathbb{X}} = \mathrm{HF}_{\mathbb{Y}}:\ 1\ 3\ 5\ 6\ 6\ \cdots$, the graded free resolutions of both coordinate rings are

$$0 \longrightarrow P(-5) \longrightarrow P(-2) \oplus P(-3) \longrightarrow P \longrightarrow R \longrightarrow 0$$

and the HF of $\Omega^1_{R_{\mathbb{X}}/K}$ and $\Omega^1_{R_{\mathbb{Y}}/K}$ agree: $0\ 3\ 8\ 11\ 10\ 7\ 6\ 6\ \cdots$

However, $\mathrm{HF}_{\Omega^2_{R_{\mathbb{X}}/K}}:\ 0\ 0\ 3\ 6\ 4\ 1\ 0\ 0\ \cdots$ and

$\mathrm{HF}_{\Omega^2_{R_{\mathbb{Y}}/K}}:\ 0\ 0\ 3\ 6\ 5\ 1\ 0\ 0\ \cdots$ differ.

Questions

(1) What is the Hilbert polynomial of $\Omega^m_{R/K}$?

(2) What is the regularity index of $\Omega^m_{R/K}$? Do we have good bounds for it?

(3) Which geometric properties of $\mathbb{X}$ can we characterize using the Hilbert functions of $\Omega^m_{R/K}$?

3. Normalization

Darth Vader: **You have learned much, young one.**

Luke: **You'll find I'm full of surprises.**

(from Star Wars - Episode V)

$\mathbb{X}$ 0-dimensional subscheme of $\mathbb{P}^n$

$R = P/I_{\mathbb{X}}$ homogeneous coordinate ring of $\mathbb{X}$

$Q^h(R) = \{ \frac{a}{b} \mid a, b \in R,\ b$ homogeneous non-zerodivisor $\}$

homogeneous quotient ring of R

Lemma

$Q^h(R) = R_{x_0}$

The Affine Coordinate Ring

By assumption, we have $\mathbb{X} \subseteq D_+(x_0) \cong \mathbb{A}^n$.

$S \cong R/\langle x_0 - 1 \rangle \cong K[x_1, \dots, x_n]/I_{\mathbb{X}}^{\mathrm{deh}}$ **affine coordinate ring** of $\mathbb{X}$

For $i \geq r_{\mathbb{X}}$, we have $R_i \cong S\, x_0^i$ via $f \mapsto f^{\mathrm{deh}}\, x_0^i$.

Theorem

(a) $Q^h(R) \cong S[x_0, x_0^{-1}]$

(b) $\widetilde{R} = S[x_0] \subseteq Q^h(R)$ *is an integral extension of* R *via* $f \mapsto f^{\mathrm{deh}}\, x_0^d$ *for* $f \in R_d$.

(c) $\widetilde{R}$ *is the integral closure of* R *in* $Q^h(R)$ *iff* $\mathbb{X}$ *is reduced.*

Theorem

Let $\widetilde{R} = S[x_0]$.

(a) $\Omega^1_{\widetilde{R}/K} = S[x_0]dx_0 \;\oplus\; K[x_0] \otimes \Omega^1_{S/K}$

(b) $\mathrm{HF}_{\Omega^1_{\widetilde{R}/K}}(0) = \dim_K(\Omega^1_{S/K})$ *and for* $i \geq 1$ *we have*

$$\mathrm{HF}_{\Omega^1_{\widetilde{R}/K}}(i) = \dim_K(\Omega^1_{S/K}) + \dim_K(S)$$

(c) The scheme $\mathbb{X}$ is reduced iff $\Omega^1_{S/K} = 0$.

4. Regularity Bounds

I don't know why the sacrifice didn't work.
The science was so solid.
(King Julien)

$\mathbb{X}$ 0-dimensional subscheme of $\mathbb{P}^n$

$R = P/I_{\mathbb{X}}$ homogeneous coordinate ring of $\mathbb{X}$

Theorem

(a) *For $i \geq 2r_{\mathbb{X}} + 1$, the multiplication by x_0 yields an isomorphism* $\mu : (\Omega^1_{R/K})_i \longrightarrow (\Omega^1_{R/K})_{i+1}$.

(b) $\mathrm{ri}(\Omega^1_{R/K}) \leq 2\,\mathrm{ri}_{\mathbb{X}} + 1$

Note that the monomorphism $\imath: R \hookrightarrow \widetilde{R} = S[x_0]$ induces a canonical R-module homomorphism $\psi: \Omega^1_{R/K} \longrightarrow \Omega^1_{\widetilde{R}/K}$.

Its kernel is the **torsion submodule** of $\Omega^1_{R/K}$, i.e., $T\Omega^1_{R/K} = \ker(\psi) = \{w \in \Omega^1_{R/K} \mid x_0^i w = 0 \text{ for some } i \geq 1\}$.

Theorem

(a) *For $i \geq 2r_{\mathbb{X}} + 1$, we have $(T\Omega^1_{R/K})_i = 0$ and*

$$\psi_i: (\Omega^1_{R/K})_i \longrightarrow (\Omega^1_{\widetilde{R}/K})_i \cong S[x_0]_{i-1}dx_0 \oplus x_0^i \cdot \Omega^1_{S/K}$$

is an isomorphism of K-vector spaces.

(b) *We have $\mathrm{HP}(\Omega^1_{R/K}) = \deg(\mathbb{X}) + \dim_K(\Omega^1_{S/K})$.*

For the ring $\widetilde{R} = S[x_0]$, we can compute $\Omega^m_{\widetilde{R}/K}$ as follows.

Theorem

(a) $\Omega^m_{\widetilde{R}/K} \cong K[x_0] \otimes \Omega^m_{S/K} \;\oplus\; K[x_0] dx_0 \wedge \Omega^{m-1}_{S/K}$

(b) *The canonical R-module homomorphism* $\Lambda^m \psi : \; \Omega^m_{R/K} \longrightarrow \Omega^m_{\widetilde{R}/K}$ *is an isomorphism in degrees* $\geq 2r_{\mathbb{X}} + m$*.*

(c) $\mathrm{HP}(\Omega^m_{R/K}) = \dim_K(\Omega^m_{S/K}) + \dim_K(\Omega^{m-1}_{S/K})$.

(d) $\mathrm{ri}(\Omega^m_{R/K}) \leq 2r_{\mathbb{X}} + m$

5. Application to Points in the Plane

Haters will see you walk on water and say:
“It’s because he can’t swim.”
(Anonymous)

$\mathbb{X} \subset \mathbb{P}^2$ finite set of s points

$R = P/I_{\mathbb{X}} = K[x_0, x_1, x_2]/I_{\mathbb{X}}$ homogeneous coordinate ring

$S = K[x_1, x_2]/I_{\mathbb{X}}^{\mathrm{deh}}$ affine coordinate ring of $\mathbb{X}$ in $D_+(x_0) \cong \mathbb{A}^2$

Example

For $s = 3$, we have $\mathrm{HF}_{\mathbb{X}}:\ 1\ 2\ 3\ 3\ \cdots$ if the three points are collinear and $\mathrm{HF}_{\mathbb{X}}:\ 1\ 3\ 3\ \cdots$ otherwise.

Example (Four Points in the Plane)

Let $s = 4$.

(a) We have $\mathrm{HF}_{\mathbb{X}}:\ 1\ 2\ 3\ 4\ 4\ \cdots$ iff the four points are collinear.

(b) Otherwise, we have $\mathrm{HF}_{\mathbb{X}}:\ 1\ 3\ 4\ 4\ \cdots$.

If the multiplication map $R_1 \otimes (\omega_R)_{-1} \longrightarrow (\omega_R)_0$ is non-degenerate then $\mathbb{X}$ is the complete intersection of two conics.

(c) Otherwise, $\mathbb{X}$ consists of three points on a line and one point off the line.

Example (Five Points in the Plane)

Let $s = 5$.

(a) $\mathbb{X}$ consists of 5 points on a line iff $\mathrm{HF}_{\mathbb{X}}:\ 1\ 2\ 3\ 4\ 5\ 5\ \cdots$.

(b) $\mathbb{X}$ consists of 4 points on a line and one point off the line iff $\mathrm{HF}_{\mathbb{X}}:\ 1\ 3\ 4\ 5\ 5\ \cdots$.

(c) Suppose that no four points of $\mathbb{X}$ are collinear. Then we have $\mathrm{HF}_{\mathbb{X}}:\ 1\ 3\ 5\ 5\ \cdots$. The set $\mathbb{X}$ is contained in the union of two lines iff $\mathrm{HF}_{\Omega^2_{R/K}}:\ 0\ 0\ 3\ 5\ 2\ 0\ \cdots$.

(d) No three points of $\mathbb{X}$ are collinear iff $\mathrm{HF}:\ 1\ 3\ 5\ 5\ \cdots$ and $\mathrm{HF}_{\Omega^2_{R/K}}:\ 0\ 0\ 3\ 5\ 1\ 0\ \cdots$. In this case $\mathbb{X}$ is contained in a unique non-singular conic.

THE END

Give a man a mask,
and he will show you his true face.
(Oscar Wilde)

Thank you very much for your attention!