Four Dimensional affine domains.

k - field   affine domains.

$A^{[n]} := A[X_1, \ldots, X_n]$

We say B is said to be cancellative if $\underline{A^{[n]} \cong B^{[n]}}$ for some $n \geq 1$, affine ring A

$\Rightarrow \underline{A \cong B}$.

Abhyankar. Eakin - Heinzer (1972) True if B is one dimensional affine domain containing $\mathbb{Q}$.

Hochster (1972) — $\dfrac{\mathbb{R}[X, Y, Z]}{(X^2+Y^2+Z^2-1)} = R = \mathbb{R}[x, y, z]$

$$0 \longrightarrow P \longrightarrow R^3 \xrightarrow{= Re_1 + Re_2 + Re_3} R \longrightarrow 0$$

$$e_1 \longrightarrow x$$
$$e_2 \longrightarrow y$$
$$e_3 \longmapsto z$$

$P \oplus R = R^3$

$P \not\cong R^2$

$B = \mathrm{Sym}_R(P)$

$B^{[1]} \cong_R R^{[3]}$

$B \not\cong_R R^{[2]}$

$\Downarrow$

$B \not\cong_R R^{[2]}$

B — four dim affine smooth domain over $\mathbb{R}$

1989. Danielewski   two dimensional smooth affine over $\mathbb{C}$

$\forall n \geq 1 \quad B_n = \dfrac{\mathbb{C}[X, Y, Z]}{(X^nY - Z^2 + 1)}$.

$\underline{B_n^{[1]} \cong B_m^{[1]}} \quad \forall\, n, m \geq 1$

$\underline{B_n \not\cong B_m}$ if $n \neq m$.

Q. Does $\exists$ a ring R such that

$$R^{[n]} \cong_k A^{[n]}$$

but $R^{[1]} \not\cong_k A^{[1]}$.

Jelonek (2009) constructed smooth examples over $\mathbb{C}$
dim ≥ 10

based on the existence of projective modules which are two stably free but not 1-stably free.

Asanuma (2018) JCA — we have constructed an example

dim $= 4$
seminormal over $\mathbb{R}$
affine.

Dubouloz (2019) — constructed smooth affine rings over $\mathbb{C}$ and dim ≥ 2

$$A^{[2]} \cong B^{[2]}$$

but $A^{[1]} \not\cong B^{[1]}$.

R, $n \geq 1$ $\quad SL_n(R) = \{ (\quad) \mid n\times n$ invertible matrices over $R \}$

$$SL_n(R) \hookrightarrow SL_{n+1}(R)$$

$$M \longmapsto \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}$$

$$SL(R) = \bigcup_{n\geq 1} SL_n(R)$$

$$SK_1(R) = \frac{SL(R)}{[SL(R), SL(R)]}$$

In the thm

$p \in SL_2(S)$ s.t its image in $SK_1(S)$ is non zero

Suslin (Thm) $\quad SK_1(k^{[n]}) = \{0\}$.

$$R = k^{[2]} \longrightarrow \frac{R}{\quad} = S$$

$$SK_1(R) \xrightarrow{(f)} SK_1(S)$$

this need not be surjective

$\exists \ p \ \in \ SL_2(S)$ . .

Recipe for constructing C.E to 2-stably isom but not 1-stably isom

Thm: k field, $k = \mathbb{R}, \mathbb{C}$

$R = k[X,Y]$, Polynomial ring in two variables over k.

$f \in R = k[X,Y]$

$S = R/(f) = \dfrac{k[X,Y]}{(f)}$

$\eta : R \longrightarrow S$ natural k-algebra surjection

$\eta_n : M_n(R) \longrightarrow M_n(S)$ induced ring homomorphism

Suppose

(i) $S^* = R^* = k^*$

(ii) There exist an invertible matrix $P \in SL_2(S)$ whose image in $SK_1(S)$ is non zero.

(iii) If $\Phi : R[U,V,Z] \longrightarrow R[U,V,Z]$ is a k-algebra automorphism such that $\Phi(f) = \lambda f$ for some $\lambda \in k^*$, then $\Phi(R) = R$.

Let $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(R)$ be s.t $\eta_2(M) = P$.

Let $D = R[U,V] = k[X,Y,U,V]$.

$A = R[U^2,V^2] + (f)R[U,V] \subseteq D$

$B = R[(\alpha U + \beta V)^2, (\gamma U + \delta V)^2] + (f)R[U,V] \subseteq D$

Then

(i) A and B are affine seminormal domains

(ii) $A^{[1]}$ ≠ $B^{[1]}$

(ii) $A \neq B$

(iii) $A^{[2]} \cong B^{[2]}$

Proof of (iii) $\underline{A^{[2]}} \cong \underline{B^{[2]}}$ $\quad p \in SL_2(S)$

$$P_1 = \underline{\begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}} \in SL_4(S) \quad p^{-1} \in SL_2(S)$$

Then by whitehead lemma

P_1 is elementary

and hence $\eta_4(L) = P_1$ for some

elementary matrix $\underline{L \in SL_4(R)}$

Let $\wedge$ be an R-algebra automorphism of the ring $\underline{R[U, V, Z, W]}$ be defined by

$$\wedge \begin{pmatrix} U \\ V \\ Z \\ W \end{pmatrix} = L \begin{pmatrix} U \\ V \\ Z \\ W \end{pmatrix}$$

$$L \begin{pmatrix} U \\ V \\ Z \\ W \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & M' \end{pmatrix} \begin{pmatrix} U \\ V \\ Z \\ W \end{pmatrix} + (f) \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{pmatrix}$$

for some $g_1, g_2, g_3, g_4 \in R[U, V, Z, W]$

Now,

$$\wedge (A[Z, W])$$

$$= \wedge \left(\left(\underline{R[U^2, V^2] + (f) R[U, V]} \right) [Z, W] \right)$$

$$= \wedge \left(\underline{R[U^2, V^2, Z, W] + (f) R[U, V, Z, W]} \right)$$

$$= R[\lambda(U)^2, \lambda(V)^2, \lambda(Z), \lambda(W)] + (f) R[U, V, Z, W]$$

$$R[(\alpha U + \beta V)^2, (\gamma U + \delta V)^2, \underline{(\alpha' Z + \beta' W)}, \underline{(\gamma' Z + \delta' W)}]$$

$= R[(\alpha U + \ldots$ $\quad + (f) R[U, V, Z, W]$

$= R[(\alpha U + \beta V)^2, (\gamma U + \delta V)^2, Z, W] + (f) R[U, V, Z, W]$

since $\eta_2(M') = P^{-1}$

$= B[Z, W]$

Ex: $k = \mathbb{R}$

$f = x^2 + y^2 - 1 \in \mathbb{R}[X, Y] = R$

$S = R/(f)$

$P = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ is not stably elementary.

$\Rightarrow$ Milnor. P is not trivial in $SK_1(S)$

Ex: $k = \mathbb{C}$

$f = y^2 - x^3 - xy \in R = \mathbb{C}[X, Y]$

$S = R/(f) \cong \mathbb{C}[T^2 - T, T^3 - T^2] \hookrightarrow \mathbb{C}[T]$

$(T^3 - T^2)^2 - (T^2 - T)^3 - (T^2 - T)(T^3 - T^2)$

$= \cancel{T^6} - 2\cancel{T^5} + \cancel{T^4}$

$- \cancel{T^6} + 3\cancel{T^5} - \cancel{3T^4} + \cancel{T^3}$

$- \cancel{T^5} + \cancel{T^4} + \cancel{T^4} - \cancel{T^3}$

$SK_1(S) = K_2(\mathbb{C})$

this is uncountable

$p \in SL_2(S)$ whose image in $SK_1(S)$ is non zero

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