

The Pick–Nevanlinna problem: from metric geometry to matrix positivity

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Eigenfunction 2019 (with Apoorva Khare)
Indian Institute of Science

April 12, 2019

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- (*) $\Omega_k \subset \mathbb{C}^{n_k}$ are domains, $k = 1, 2$. Given M distinct points $z_1, \dots, z_M \in \Omega_1$, and points w_1, \dots, w_M in Ω_2 , find nec. & suff. conditions on

$$(z_1, w_1), (z_2, w_2), \dots, (z_M, w_M),$$

such that there exists a holomorphic map $F : \Omega_1 \rightarrow \Omega_2$ satisfying $F(z_j) = w_j$, $1 \leq j \leq M$.

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Theorem (G. Pick, R. Nevanlinna).

Let z_1, \dots, z_M be distinct points in \mathbb{D} and $w_1, \dots, w_M \in \mathbb{D}$. There exists $F \in \text{Hol}(\mathbb{D}; \mathbb{D})$ satisfying $F(z_j) = w_j$, $1 \leq j \leq M$, iff the matrix

$$\left[\frac{1 - \bar{w}_k w_j}{1 - \bar{z}_k z_j} \right]_{j,k=1}^M$$

is positive semi-definite.

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 - ▶ \mathbf{eval}_x is a bounded linear functional $\forall x \in S$.
- Equip \mathbb{C}^n with a complex inner product. Define the vector space

$$\text{Mult}(\mathcal{H}, \mathcal{H} \otimes \mathbb{C}^n) := \{\phi : S \rightarrow \mathbb{C}^n \mid h\phi \in \mathcal{H} \otimes \mathbb{C}^n \ \forall h \in \mathcal{H}\}$$

viewed as a subspace of $\mathcal{B}(\mathcal{H}, \mathcal{H} \otimes \mathbb{C}^n)$.

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- If we write $M_\phi(h) := h \otimes \phi (= h\phi)$, $h \in \mathcal{H}$, then it's **easy** to show:

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With these constructs, we discover...

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Proposition (Sarason).

Let S be a non-empty set and \mathcal{H} a Hilbert function space on it. Fix $\langle \cdot, \cdot \rangle$ on \mathbb{C}^n . Let x_1, \dots, x_M be distinct points in S and $w_1, \dots, w_M \in \mathbb{C}^n$ s.t. $\|w_j\| \leq 1$, $1 \leq j \leq M$.

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$$\begin{aligned} \langle g, M_\phi^* K(\cdot, x) \rangle &= \langle \phi g, K(\cdot, x) \rangle = \phi(x) g(x) = \phi(x) \langle g, K(\cdot, x) \rangle \quad \forall g \in \mathcal{H} \\ &\Rightarrow M_\phi^* K(\cdot, x) = \overline{\phi(x)} K(\cdot, x). \end{aligned}$$

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Testing the positivity of $(\mathbb{I} - M_\phi M_\phi^*)$ on the vector

$$h := \sum_{k=1}^M \bar{v}_j K(\cdot, x_j) \in \mathcal{H},$$

where $v_1, \dots, v_M \in \mathbb{C}$ gives...

4 Towards a necessary condition: Positivity

$$\begin{aligned} & \langle (\mathbb{I} - M_\phi M_\phi^*) \sum_{k=1}^M \bar{v}_k K(\cdot, x_k), \sum_{j=1}^M \bar{v}_j K(\cdot, x_j) \rangle \\ &= \langle \sum_{k=1}^M (\bar{v}_k - (\bar{v}_k \phi) \bar{w}_k) K(\cdot, x_k), \sum_{j=1}^M \bar{v}_j K(\cdot, x_j) \rangle \geq 0 \end{aligned}$$

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Important remark: If $n \geq 2$ and $\langle \cdot, \cdot \rangle$ is the standard complex inner product, then the conclusion of the above is that a certain matrix consisting of M^2 $n \times n$ blocks is positive semi-definite that implies:

$$[(1 - \langle w_j, w_k \rangle) K(x_j, x_k)]_{j,k=1}^M$$

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5 A Pick–Nevanlinna interpolation theorem

The nec. cond'n. for a contractive multiplier in $\text{Mult}(\mathcal{H}, \mathcal{H} \otimes \mathbb{C}^n)$ that interpolates allows us to prove a theorem of which the Pick–Nevanlinna result is a special case.

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Theorem.

Let z_1, \dots, z_M be distinct points in \mathbb{D} and $w_1, \dots, w_M \in \mathbb{B}^n$. There exists $F \in \text{Hol}(\mathbb{D}; \mathbb{B}^n)$ satisfying $F(z_j) = w_j$, $1 \leq j \leq M$, iff the matrix

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Establishing the necessary cond'n. We define the Hardy space $H^2(\mathbb{D})$:

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This is a Hilbert function space for which $K(\cdot, x) = 1/(1 - \bar{x}\cdot)$ for any $x \in \mathbb{D}$ and — taking the standard inner product on \mathbb{C}^n :

$\text{Mult}(\mathcal{H}, \mathcal{H} \otimes \mathbb{C}^n) \supseteq \{ \phi : \mathbb{D} \rightarrow \mathbb{C}^n \mid \phi \text{ is holo. \& bounded} \}$, **[they are actually equal]**

$$\|M_{\phi}\|_{\text{op}} = \sup_{z \in \mathbb{D}} \|\phi(z)\|.$$

6 Necessity of positivity

Thus, to find a necessary condition for a \mathbb{B}^n -valued *holomorphic* function mapping z_j to w_j , $1 \leq j \leq M$, one views the latter as sitting in $\text{Mult}(\mathcal{H}, \mathcal{H} \otimes \mathbb{C}^n)$ and applies the Sarason(–Nevanlinna) proposition with:

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7 Geometry: transitive action

For $a \in \mathbb{B}^n \setminus \{0\}$ let proj_a be the orthogonal projection onto $\text{span}_{\mathbb{C}}\{a\}$ and $Q_a = \text{id}_{\mathbb{C}^n} - \text{proj}_a$. Then

$$\Psi_a(z) := \frac{a - \text{proj}_a(z) - (1 - \|a\|^2)^{1/2} Q_a(z)}{1 - \langle z, a \rangle} \quad \forall z \in \mathbb{B}^n$$

is a holomorphic $\mathbb{B}^n \rightarrow \mathbb{B}^n$ map with the properties:

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In particular, $\Psi_a \in \text{Aut}(\mathbb{B}^n) \quad \forall a \in \mathbb{B}^n \setminus \{0\}$.

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A very special case of this map is when $n = 1$, in which case we shall denote the map by ψ_a .

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In particular, $\Psi_a \in \text{Aut}(\mathbb{B}^n) \quad \forall a \in \mathbb{B}^n \setminus \{0\}$.

A very special case of this map is when $n = 1$, in which case we shall denote the map by ψ_a . In this case, we get the (**very** familiar) map:

$$\psi_a(z) := \frac{a - z}{1 - \bar{a}z} \quad \forall z \in \mathbb{D}.$$

8 Some metric geometry

The Kobayashi (pseudo)distance: Given a domain $\Omega \subset \mathbb{C}^n$, the *Kobayashi pseudodistance* on Ω is:

$$K_{\Omega}(z, w) := \inf_{\mathcal{C}} \left\{ \sum_{j=1}^N \rho_{\mathbb{D}}(0, \zeta_j) : (f_1, \dots, f_N; \zeta_1, \dots, \zeta_N) \in \mathcal{C} \right\}$$

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where \mathcal{C} is the collection of all chains $(f_1, \dots, f_N; \zeta_1, \dots, \zeta_N)$ of $\mathbb{D} \rightarrow \Omega$ analytic discs linking z to w : i.e., $f_1(0) = z$, $f_N(\zeta_N) = w$ and $f_j(\zeta_j) = f_{j+1}(0)$, $1 \leq j \leq N - 1$.

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Obvious from construction (since the composition of holo. maps is holo.) that given domains Ω_1, Ω_2 and $\phi \in \text{Hol}(\Omega_1; \Omega_2)$,

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Examples: Both these are relevant to us:

$$K_{\mathbb{D}}(z_1, z_2) = \rho_{\mathbb{D}}(z_1, z_2) = \tanh^{-1} \left| \frac{z_1 - z_2}{1 - \bar{z}_2 z_1} \right|,$$

$$K_{\mathbb{B}^n}(w_1, w_2) = \tanh^{-1} \frac{(\|w_1\|^2 + \|w_2\|^2 - 2\text{Re}\langle w_1, w_2 \rangle + (|\langle w_1, w_2 \rangle|^2 - \|w_1\|^2 \|w_2\|^2))^{1/2}}{|1 - \langle w_1, w_2 \rangle|}.$$

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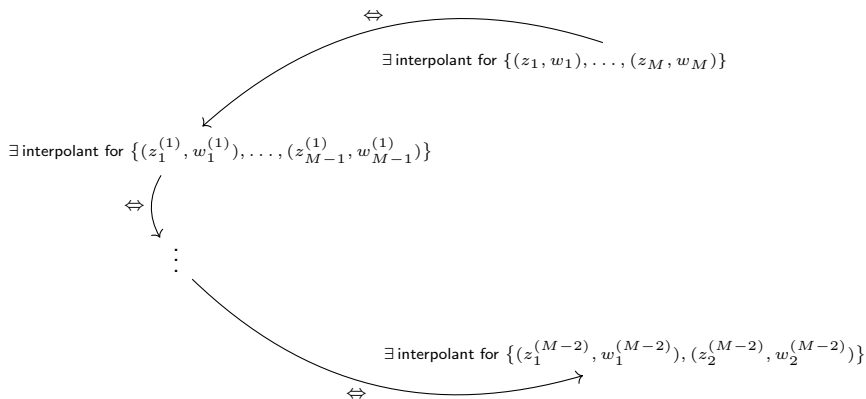
$\forall z \in \mathbb{D} \setminus \{0\}$, so $F^\bullet \in \mathcal{B}$!

¹⁰The first step towards proving sufficiency of positivity

The deflation trick reduces our problem to that of characterising existence of a 2-point interpolant:

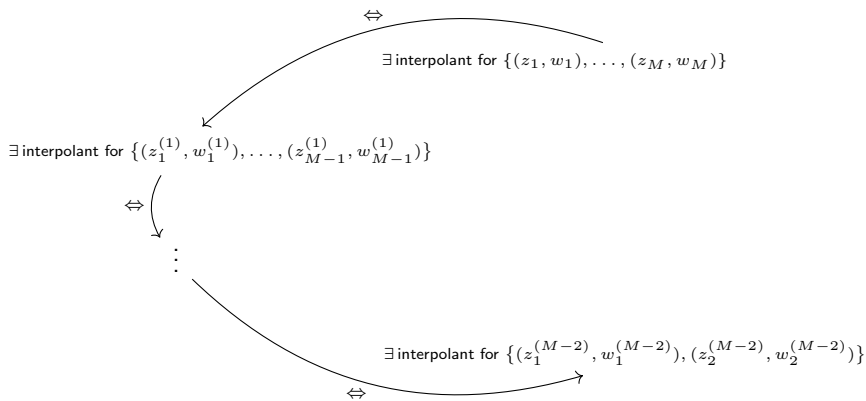
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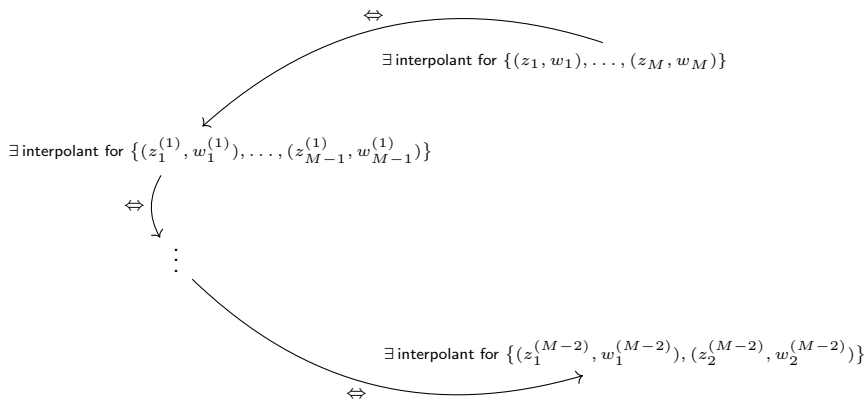
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Suppose $\{(a_1, B_1), (a_2, B_2)\}$, $a_j \in \mathbb{D}$, $B_j \in \mathbb{B}^n$ satisfy

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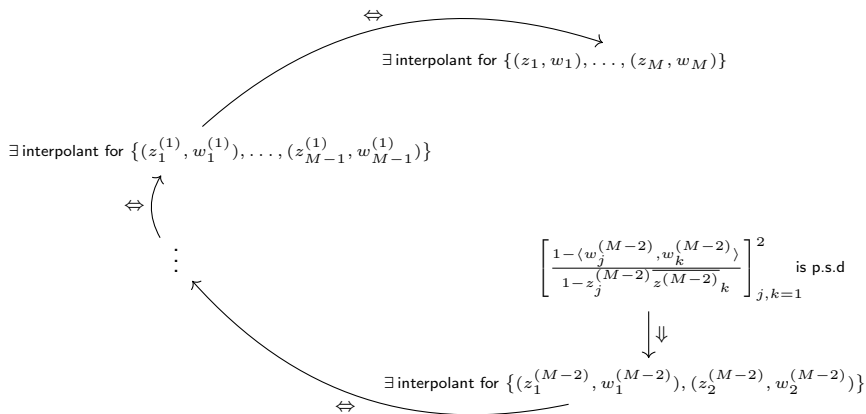
$$\left[\frac{1 - \langle b_j, b_k \rangle}{1 - a_j \bar{a}_k} \right]_{j,k=1}^2 !$$

¹¹The link to a positive semi-definite matrix

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¹²Sufficiency of positivity

Establishing the sufficient cond'n. Let a_1, \dots, a_N be distinct points in \mathbb{D} , and let b_1, \dots, b_N be points in \mathbb{B}^n , $N \geq 3$. Consider the two Hermitian forms:

$$H[a_1, \dots, a_N; b_1, \dots, b_N](\xi) := \sum_{j,k=1}^N \frac{1 - \langle b_j, b_k \rangle}{1 - a_j \bar{a}_k} \xi_j \bar{\xi}_k \quad \text{on } \mathbb{C}^N,$$

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$$\frac{1 - a_j^{(1)} \overline{a^{(1)}_k}}{1 - a_j \bar{a}_k} = \frac{\sqrt{1 - |a_N|^2}}{1 - \bar{a}_N a_j} \frac{\sqrt{1 - |a_N|^2}}{1 - a_N \bar{a}_k} \equiv \alpha_j \bar{\alpha}_k,$$
$$\frac{1 - \langle b_j^{(1)}, b_k^{(1)} \rangle}{1 - \langle b_j, b_k \rangle} = \frac{\sqrt{1 - \|b_N\|^2}}{1 - \langle b_j, b_N \rangle} \frac{\sqrt{1 - \|b_N\|^2}}{1 - \langle b_N, b_k \rangle} \equiv \beta_j \bar{\beta}_k.$$

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It would suffice to prove that

$H[a_1, \dots, a_N; b_1, \dots, b_N] \geq 0 \Rightarrow \tilde{H}[a_1, \dots, a_N; b_1, \dots, b_N] \geq 0$ to establish our theorem. We compute:

$$\frac{1 - a_j^{(1)} \overline{a^{(1)}_k}}{1 - a_j \bar{a}_k} = \frac{\sqrt{1 - |a_N|^2}}{1 - \bar{a}_N a_j} \frac{\sqrt{1 - |a_N|^2}}{1 - a_N \bar{a}_k} \equiv \alpha_j \bar{\alpha}_k,$$
$$\frac{1 - \langle b_j^{(1)}, b_k^{(1)} \rangle}{1 - \langle b_j, b_k \rangle} = \frac{\sqrt{1 - \|b_N\|^2}}{1 - \langle b_j, b_N \rangle} \frac{\sqrt{1 - \|b_N\|^2}}{1 - \langle b_N, b_k \rangle} \equiv \beta_j \bar{\beta}_k.$$

This computation gives...

13 Sufficiency of positivity, cont'd.

$$\begin{aligned} H[a_1^{(1)}, \dots, a_{N-1}^{(1)}, 0; b_1^{(1)}, \dots, b_{N-1}^{(1)}, 0](\xi) \\ = H[a_1, \dots, a_N; b_1, \dots, b_N](\text{Diag}(\beta_1/\alpha_1, \dots, \beta_N/\alpha_N) \xi) \end{aligned}$$

13 Sufficiency of positivity, cont'd.

$$\begin{aligned} H[a_1^{(1)}, \dots, a_{N-1}^{(1)}, 0; b_1^{(1)}, \dots, b_{N-1}^{(1)}, 0](\xi) \\ = H[a_1, \dots, a_N; b_1, \dots, b_N](\text{Diag}(\beta_1/\alpha_1, \dots, \beta_N/\alpha_N) \xi) \end{aligned}$$

Hence, the form on the L.H.S. is non-negative if $H[a_1, \dots, a_N; b_1, \dots, b_N] \geq 0$.

13 Sufficiency of positivity, cont'd.

$$\begin{aligned} H[a_1^{(1)}, \dots, a_{N-1}^{(1)}, 0; b_1^{(1)}, \dots, b_{N-1}^{(1)}, 0](\xi) \\ = H[a_1, \dots, a_N; b_1, \dots, b_N](\text{Diag}(\beta_1/\alpha_1, \dots, \beta_N/\alpha_N) \xi) \end{aligned}$$

Hence, the form on the L.H.S. is non-negative if $H[a_1, \dots, a_N; b_1, \dots, b_N] \geq 0$.

We now invoke the following:

Result (Schur). *Let \mathcal{K} be a complex inner-product space with inner product $(\cdot | \cdot)$. Let $c_1, \dots, c_{N-1} \in \mathbb{D} \setminus \{0\}$ and set $c_N := 0$. Let $B_1, \dots, B_{N-1} \in \mathcal{K}$ with $\|B_j\|_{\mathcal{K}} < 1$ and set $B_N := 0$. If the quadratic form*

$$Q(\xi) := \sum_{j,k=1}^N \frac{1 - (B_j | B_k)}{1 - c_j \bar{c}_k} \xi_j \bar{\xi}_k \quad \text{on } \mathbb{C}^N$$

is conditionally positive,

13 Sufficiency of positivity, cont'd.

$$\begin{aligned} H[a_1^{(1)}, \dots, a_{N-1}^{(1)}; 0; b_1^{(1)}, \dots, b_{N-1}^{(1)}](\xi) \\ = H[a_1, \dots, a_N; b_1, \dots, b_N](\text{Diag}(\beta_1/\alpha_1, \dots, \beta_N/\alpha_N) \xi) \end{aligned}$$

Hence, the form on the L.H.S. is non-negative if $H[a_1, \dots, a_N; b_1, \dots, b_N] \geq 0$. We now invoke the following:

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is conditionally positive, then the quadratic form

$$\tilde{Q}(\xi) := \sum_{j,k=1}^{N-1} \frac{1 - (c_j^{-1} B_j | c_k^{-1} B_k)}{1 - c_j \bar{c}_k} \xi_j \bar{\xi}_k \quad \text{on } \mathbb{C}^{N-1}$$

is positive semi-definite on \mathbb{C}^{N-1} .

13 Sufficiency of positivity, cont'd.

$$\begin{aligned} H[a_1^{(1)}, \dots, a_{N-1}^{(1)}, 0; b_1^{(1)}, \dots, b_{N-1}^{(1)}, 0](\xi) \\ = H[a_1, \dots, a_N; b_1, \dots, b_N](\text{Diag}(\beta_1/\alpha_1, \dots, \beta_N/\alpha_N) \xi) \end{aligned}$$

Hence, the form on the L.H.S. is non-negative if $H[a_1, \dots, a_N; b_1, \dots, b_N] \geq 0$. We now invoke the following:

Result (Schur). Let \mathcal{K} be a complex inner-product space with inner product $(\cdot | \cdot)$. Let $c_1, \dots, c_{N-1} \in \mathbb{D} \setminus \{0\}$ and set $c_N := 0$. Let $B_1, \dots, B_{N-1} \in \mathcal{K}$ with $\|B_j\|_{\mathcal{K}} < 1$ and set $B_N := 0$. If the quadratic form

$$Q(\xi) := \sum_{j,k=1}^N \frac{1 - (B_j | B_k)}{1 - c_j \bar{c}_k} \xi_j \bar{\xi}_k \quad \text{on } \mathbb{C}^N$$

is conditionally positive, then the quadratic form

$$\tilde{Q}(\xi) := \sum_{j,k=1}^{N-1} \frac{1 - (c_j^{-1} B_j | c_k^{-1} B_k)}{1 - c_j \bar{c}_k} \xi_j \bar{\xi}_k \quad \text{on } \mathbb{C}^{N-1}$$

is positive semi-definite on \mathbb{C}^{N-1} .

Just set $c_j = a_j^{(1)}$ and $B_j = b_j^{(1)}$, $1 \leq j \leq N$, and we're done! ■