The Pick–Nevanlinna problem: from metric geometry to matrix positivity

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Eigenfunction 2019 (with Apoorva Khare)

Indian Institute of Science

April 12, 2019

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$$(z_1, w_1),\ (z_2, w_2), \dots, (z_M, w_M),$$

such that there exists a holomorphic map $F : \Omega_1 \to \Omega_2$ satisfying $F(z_j) = w_j$, $1 \leq j \leq M$.

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Theorem (G. Pick, R. Nevanlinna).
Let $z_1, \dots, z_M$ be distinct points in $\mathbb{D}$ and $w_1, \dots, w_M \in \mathbb{D}$. There exists $F \in \mathrm{Hol}(\mathbb{D}; \mathbb{D})$ satisfying $F(z_j) = w_j$, $1 \le j \le M$, iff the matrix

$$\left[\frac{1 - \overline{w}_k w_j}{1 - \overline{z}_k z_j}\right]_{j,k=1}^{M}$$

is positive semi-definite.

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- Equip $\mathbb{C}^n$ with a complex inner product. Define the vector space

$$\mathrm{Mult}(\mathcal{H}, \mathcal{H} \otimes \mathbb{C}^n) := \{\phi : S \to \mathbb{C}^n \mid h\phi \in \mathcal{H} \otimes \mathbb{C}^n \ \forall h \in \mathcal{H}\}$$

  viewed as a subspace of $\mathcal{B}(\mathcal{H}, \mathcal{H} \otimes \mathbb{C}^n)$.

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- If we write $M_\phi(h) := h \otimes \phi \, (= h\phi)$, $h \in \mathcal{H}$, then it's **easy** to show:

$$\|M_\phi\|_{\mathsf{op}} \leq 1 \iff (\mathbb{I} - M_\phi M_\phi^*) \text{ is positive semi-definite.}$$

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With these constructs, we discover…

$_3$Towards a necessary condition, cont'd.

Proposition (Sarason)**.**
Let S be a non-empty set and $\mathcal{H}$ a Hilbert function space on it. Fix $\langle \cdot\,, \cdot \rangle$ on $\mathbb{C}^n$. Let $x_1, \dots, x_M$ be distinct points in S and $w_1, \dots, w_M \in \mathbb{C}^n$ s.t. $\|w_j\| \leq 1$, $1 \leq j \leq M$.

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$$\langle\, g, M_\phi^* K(\cdot\,, x) \rangle = \langle \phi g, K(\cdot\,, x) \rangle = \phi(x) g(x) = \phi(x) \langle\, g, K(\cdot\,, x) \rangle \ \ \forall g \in \mathcal{H}$$

$$\Rightarrow \ M_\phi^* K(\cdot\,, x) = \overline{\phi(x)} K(\cdot\,, x).$$

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Testing the positivity of $(\mathbb{I}-M_\phi M_\phi^*)$ on the vector

$$h := \textstyle\sum_{k=1}^{M} \overline{v}_j K(\cdot\,,x_j)\in\mathcal{H},$$

where $v_1,\dots,v_M\in\mathbb{C}$ gives...

Towards a necessary condition: Positivity

$$\langle (\mathbb{I} - M_\phi M_\phi^*) \textstyle\sum_{k=1}^{M} \overline{v}_k K(\cdot\,, x_k),\ \sum_{j=1}^{M} \overline{v}_j K(\cdot\,, x_j) \rangle$$

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Important remark: If $n \geq 2$ and $\langle \cdot\,, \cdot \rangle$ is the standard complex inner product, then the conclusion of the above is that a certain matrix consisting of M^2 $n \times n$ blocks is positive semi-definite that implies:

$$\big[\big(1 - \langle w_j, w_k \rangle\big) K(x_j, x_k)\big]_{j,k=1}^M$$

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$_5$A Pick–Nevanlinna interpolation theorem

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Theorem.
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Establishing the necessary cond'n. We define the *Hardy space* $H^2(\mathbb{D})$:

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This is a Hilbert function space for which $K(\cdot\,, x) = 1/(1 - \overline{x}\,\cdot)$ for any $x \in \mathbb{D}$ and — taking the standard inner product on $\mathbb{C}^n$:

$$\mathrm{Mult}(\mathcal{H}, \mathcal{H} \otimes \mathbb{C}^n) \supseteq \{\phi : \mathbb{D} \to \mathbb{C}^n \mid \phi \text{ is holo. \& bounded}\},$$ **[they are actually equal]**

$$\|M_\phi\|_{\mathsf{op}} = \sup_{z \in \mathbb{D}} \|\phi(z)\|.$$

$_6$Necessity of positivity

Thus, to find a necessary condition for a $\mathbb{B}^n$-valued *holomorphic* function mapping z_j to w_j, $1 \leq j \leq M$, one views the latter as sitting in $\mathrm{Mult}(\mathcal{H}, \mathcal{H} \otimes \mathbb{C}^n)$ and applies the Sarason(–Nevanlinna) proposition with:

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is positive semi-definite. ∎

Geometry: transitive action

For $a \in \mathbb{B}^n \setminus \{0\}$ let proj_a be the orthogonal projection onto $\mathrm{span}_{\mathbb{C}}\{a\}$ and $Q_a = \mathsf{id}_{\mathbb{C}^n} - \mathsf{proj}_a$. Then

$$\Psi_a(z) := \frac{a - \mathsf{proj}_a(z) - (1 - \|a\|^2)^{1/2} Q_a(z)}{1 - \langle z, a \rangle} \quad \forall z \in \mathbb{B}^n$$

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In particular, $\Psi_a \in \mathrm{Aut}(\mathbb{B}^n)$ $\forall a \in \mathbb{B}^n \setminus \{0\}$.

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A very special case of this map is when $n = 1$, in which case we shall denote the map by ψ_a.

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$$\Psi_a(z) := \frac{a - \mathsf{proj}_a(z) - (1 - \|a\|^2)^{1/2} Q_a(z)}{1 - \langle z, a \rangle} \ \ \forall z \in \mathbb{B}^n$$

is a holomorphic $\mathbb{B}^n \to \mathbb{B}^n$ map with the properties:

- $\Psi_a(a) = 0$ and $\Psi_a(0) = 0$,
- $\Psi_a \circ \Psi_a = \mathsf{id}_{\mathbb{B}^n}$.

In particular, $\Psi_a \in \mathrm{Aut}(\mathbb{B}^n) \ \forall a \in \mathbb{B}^n \setminus \{0\}$.

A very special case of this map is when $n = 1$, in which case we shall denote the map by ψ_a. In this case, we get the (**very** familiar) map:

$$\psi_a(z) := \frac{a - z}{1 - \overline{a}z} \ \ \forall z \in \mathbb{D}.$$

Some metric geometry

The Kobayashi (pseudo)distance: Given a domain $\Omega \subset \mathbb{C}^n$, the *Kobayashi pseudodistance* on Ω is:

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where $\mathscr{C}$ is the collection of all chains $(f_1,\dots,f_N;\zeta_1,\dots\zeta_N)$ of $\mathbb{D} \to \Omega$ analytic discs linking z to w: i.e., $f_1(0) = z$, $f_N(\zeta_N) = w$ and $f_j(\zeta_j) = f_{j+1}(0)$, $1 \leq j \leq N-1$.

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Obvious from construction (since the composition of holo. maps is holo.) that given domains Ω_1, Ω_2 and $\phi \in \mathrm{Hol}(\Omega_1;\Omega_2)$,

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Examples: Both these are relevant to us:

$$K_{\mathbb{D}}(z_1,z_2) = \rho_{\mathbb{D}}(z_1,z_2) = \tanh^{-1}\left|\frac{z_1 - z_2}{1-\overline{z}_2 z_1}\right|,$$

$$K_{\mathbb{B}^n}(w_1,w_2) = \tanh^{-1} \frac{\big(\|w_1\|^2 + \|w_2\|^2 - 2\mathrm{Re}\langle w_1,w_2\rangle + (\,|\langle w_1,w_2\rangle|^2 - \|w_1\|^2\|w_2\|^2)\big)^{1/2}}{|1-\langle w_1,w_2\rangle|}.$$

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A key conclusion from the last slide

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Here, $F^\bullet := z^{-1}\widetilde{F}(z)$. Its holomorphicity would follow from Riemann's RST if we can show that $F^\bullet|_{\mathbb{D}\setminus\{0\}}$ is bounded. By above inequality, $\|\widetilde{F}(z)\|/|z| \leq 1$ $\forall z \in \mathbb{D}\setminus\{0\}$, so $F^\bullet \in \mathscr{B}$!

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The link to a positive semi-definite matrix

Suppose $\{(a_1, B_1), (a_2, B_2)\}$, $a_j \in \mathbb{D}$, $B_j \in \mathbb{B}^n$ satisfy
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$$\left[\frac{1 - \langle b_j, b_k\rangle}{1 - a_j\overline{a}_k}\right]_{j,k=1}^2!$$

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$$\left[\frac{1-\langle w_j^{(M-2)}, w_k^{(M-2)}\rangle}{1-z_j^{(M-2)}\overline{z^{(M-2)}}_k}\right]^2_{j,k=1} \text{ is p.s.d}$$

$$\Downarrow$$

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12 Sufficiency of positivity

Establishing the sufficient cond'n. Let $a_1, \dots, a_N$ be distinct points in $\mathbb{D}$, and let $b_1, \dots, b_N$ be points in $\mathbb{B}^n$, $N \geq 3$. Consider the two Hermitian forms:

$$H[a_1, \dots, a_N; b_1, \dots, b_N](\xi) := \sum_{j,k=1}^{N} \frac{1 - \langle b_j, b_k \rangle}{1 - a_j \overline{a}_k} \xi_j \overline{\xi}_k \quad \text{on } \mathbb{C}^N,$$

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It would suffice to prove that $H[a_1, \dots, a_N; b_1, \dots, b_N] \geq 0 \Rightarrow \widetilde{H}[a_1, \dots, a_N; b_1, \dots, b_N] \geq 0$ to establish our theorem. We compute:

$$\frac{1 - a_j^{(1)} \overline{a^{(1)}}_k}{1 - a_j \overline{a}_k} = \frac{\sqrt{1 - |a_N|^2}}{1 - \overline{a}_N a_j} \frac{\sqrt{1 - |a_N|^2}}{1 - a_N \overline{a}_k} \equiv \alpha_j \overline{\alpha}_k,$$

$$\frac{1 - \langle b_j^{(1)}, b_k^{(1)} \rangle}{1 - \langle b_j, b_k \rangle} = \frac{\sqrt{1 - \|b_N\|^2}}{1 - \langle b_j, b_N \rangle} \frac{\sqrt{1 - \|b_N\|^2}}{1 - \langle b_N, b_k \rangle} \equiv \beta_j \overline{\beta}_k.$$

Sufficiency of positivity

Establishing the sufficient cond'n. Let $a_1, \dots, a_N$ be distinct points in $\mathbb{D}$, and let $b_1, \dots, b_N$ be points in $\mathbb{B}^n$, $N \geq 3$. Consider the two Hermitian forms:

$$H[a_1, \dots, a_N; b_1, \dots, b_N](\xi) := \sum_{j,k=1}^{N} \frac{1 - \langle b_j, b_k \rangle}{1 - a_j \overline{a}_k} \xi_j \overline{\xi}_k \quad \text{on } \mathbb{C}^N,$$

$$\widetilde{H}[a_1, \dots, a_N; b_1, \dots, b_N](\xi) := \sum_{j,k=1}^{N-1} \frac{1 - \langle \psi_{a_N}(a_j)^{-1} \Psi_{b_N}(b_j), \psi_{a_N}(a_k)^{-1} \Psi_{b_N}(b_k) \rangle}{1 - \psi_{a_N}(a_j) \overline{\psi_{a_N}(a_k)}} \xi_j \overline{\xi}_k$$

$$\text{on } \mathbb{C}^{N-1}.$$

It would suffice to prove that $H[a_1, \dots, a_N; b_1, \dots, b_N] \geq 0 \Rightarrow \widetilde{H}[a_1, \dots, a_N; b_1, \dots, b_N] \geq 0$ to establish our theorem. We compute:

$$\frac{1 - a_j^{(1)} \overline{a^{(1)}}_k}{1 - a_j \overline{a}_k} = \frac{\sqrt{1 - |a_N|^2}}{1 - \overline{a}_N a_j} \frac{\sqrt{1 - |a_N|^2}}{1 - a_N \overline{a}_k} \equiv \alpha_j \overline{\alpha}_k,$$

$$\frac{1 - \langle b_j^{(1)}, b_k^{(1)} \rangle}{1 - \langle b_j, b_k \rangle} = \frac{\sqrt{1 - \|b_N\|^2}}{1 - \langle b_j, b_N \rangle} \frac{\sqrt{1 - \|b_N\|^2}}{1 - \langle b_N, b_k \rangle} \equiv \beta_j \overline{\beta}_k.$$

This computation gives...

Sufficiency of positivity, cont'd.

$$H[a_1^{(1)}, \dots, a_{N-1}^{(1)}, 0; b_1^{(1)}, \dots, b_{N-1}^{(1)}, 0](\xi)$$
$$= H[a_1, \dots, a_N; b_1, \dots, b_N]\big(\operatorname{Diag}(\beta_1/\alpha_1, \dots, \beta_N/\alpha_N)\,\xi\,\big)$$

13 Sufficiency of positivity, cont'd.

$$H[a_1^{(1)},\dots,a_{N-1}^{(1)},0;b_1^{(1)},\dots,b_{N-1}^{(1)},0](\xi) \\ = H[a_1,\dots,a_N;b_1,\dots,b_N]\big(\,\mathrm{Diag}(\beta_1/\alpha_1,\dots,\beta_N/\alpha_N)\,\xi\,\big)$$

Hence, the form on the L.H.S. is non-negative if $H[a_1,\dots,a_N;b_1,\dots,b_N] \geq 0$.

Sufficiency of positivity, cont'd.

$$H[a_1^{(1)}, \dots, a_{N-1}^{(1)}, 0; b_1^{(1)}, \dots, b_{N-1}^{(1)}, 0](\xi) = H[a_1, \dots, a_N; b_1, \dots, b_N]\big(\,\mathrm{Diag}(\beta_1/\alpha_1, \dots, \beta_N/\alpha_N)\,\xi\,\big)$$

Hence, the form on the L.H.S. is non-negative if $H[a_1, \dots, a_N; b_1, \dots, b_N] \geq 0$. We now invoke the following:

Result (Schur). *Let $\mathcal{K}$ be a complex inner-product space with inner product $(\cdot \mid \cdot)$. Let $c_1, \dots, c_{N-1} \in \mathbb{D} \setminus \{0\}$ and set $c_N := 0$. Let $B_1, \dots, B_{N-1} \in \mathcal{K}$ with $\|B_j\|_{\mathcal{K}} < 1$ and set $B_N := 0$. If the quadratic form*

$$Q(\xi) := \sum_{j,k=1}^{N} \frac{1 - (B_j \mid B_k)}{1 - c_j \bar{c}_k} \xi_j \bar{\xi}_k \quad \text{on } \mathbb{C}^N$$

is conditionally positive,

Sufficiency of positivity, cont'd.

$$H[a_1^{(1)}, \dots, a_{N-1}^{(1)}, 0; b_1^{(1)}, \dots, b_{N-1}^{(1)}, 0](\xi) = H[a_1, \dots, a_N; b_1, \dots, b_N]\big(\,\mathrm{Diag}(\beta_1/\alpha_1, \dots, \beta_N/\alpha_N)\,\xi\,\big)$$

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is conditionally positive, then the quadratic form

$$\widetilde{Q}(\xi) := \sum_{j,k=1}^{N-1} \frac{1 - (c_j^{-1}B_j \mid c_k^{-1}B_k)}{1 - c_j\bar{c}_k} \xi_j \bar{\xi}_k \quad \text{on } \mathbb{C}^{N-1}$$

is positive semi-definite on $\mathbb{C}^{N-1}$.

Sufficiency of positivity, cont'd.

$$H[a_1^{(1)}, \dots, a_{N-1}^{(1)}, 0; b_1^{(1)}, \dots, b_{N-1}^{(1)}, 0](\xi) = H[a_1, \dots, a_N; b_1, \dots, b_N]\big(\operatorname{Diag}(\beta_1/\alpha_1, \dots, \beta_N/\alpha_N)\,\xi\,\big)$$

Hence, the form on the L.H.S. is non-negative if $H[a_1, \dots, a_N; b_1, \dots, b_N] \geq 0$. We now invoke the following:

Result (Schur). *Let $\mathcal{K}$ be a complex inner-product space with inner product $(\cdot \mid \cdot)$. Let $c_1, \dots, c_{N-1} \in \mathbb{D}\setminus\{0\}$ and set $c_N := 0$. Let $B_1, \dots, B_{N-1} \in \mathcal{K}$ with $\|B_j\|_{\mathcal{K}} < 1$ and set $B_N := 0$. If the quadratic form*

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$$\widetilde{Q}(\xi) := \sum_{j,k=1}^{N-1} \frac{1 - (c_j^{-1} B_j \mid c_k^{-1} B_k)}{1 - c_j \overline{c}_k} \xi_j \overline{\xi}_k \quad \text{on } \mathbb{C}^{N-1}$$

is positive semi-definite on $\mathbb{C}^{N-1}$.

Just set $c_j = a_j^{(1)}$ and $B_j = b_j^{(1)}$, $1 \leq j \leq N$, and we're done! ∎