

**NATIONAL BOARD FOR HIGHER MATHEMATICS**

**Research Scholarships Screening Test**

**Saturday, January 22, 2011**

**Time Allowed: 150 Minutes**

**Maximum Marks: 40**

**Please read, carefully, the instructions on the following page**

## INSTRUCTIONS TO CANDIDATES

- Please ensure that this booklet contains 11 numbered (and printed) pages. The back of each printed page is blank and can be used for rough work.
- There are **five** sections, containing **ten** questions each, entitled Algebra, Analysis, Topology, Applied Mathematics and Miscellaneous. Answer as many questions as possible. The assessment of the paper will be based on the best **four** sections. Each question carries one point and the maximum possible score is **forty**.
- Answer each question, as directed, in the space provided in the **answer booklet**, which is being supplied separately. This question paper is meant to be retained by you and so **do not** answer questions on it.
- In certain questions you are required to pick out the qualifying statement(s) from multiple choices. None of the statements, or one or more than one statement may qualify. Write **none** if none of the statements qualify, or list the labels of *all* the qualifying statements (amongst (a), (b), and (c)).
- Points will be awarded in the above questions only if **all** the correct choices are made. **There will be no partial credit.**
- $\mathbb{N}$  denotes the set of natural numbers,  $\mathbb{Z}$  - the integers,  $\mathbb{Q}$  - the rationals,  $\mathbb{R}$  - the reals and  $\mathbb{C}$  - the field of complex numbers.  $\mathbb{R}^n$  denotes the  $n$ -dimensional Euclidean space, which is assumed to be endowed with its ‘usual’ topology. The symbol  $]a, b[$  will stand for the open interval  $\{x \in \mathbb{R} \mid a < x < b\}$  while  $[a, b]$  will stand for the corresponding closed interval;  $[a, b[$  and  $]a, b]$  will stand for the corresponding left-closed-right-open and left-open-right-closed intervals respectively. The symbol  $I$  will denote the identity matrix of appropriate order. The space of continuous real valued functions on an interval  $[a, b]$  is denoted by  $\mathcal{C}[a, b]$  and is endowed with its usual ‘sup’ norm.

## Section 1: Algebra

**1.1** Solve:

$$x^4 - 3x^3 + 4x^2 - 3x + 1 = 0.$$

**1.2** Pick out the true statements:

a. Let  $H$  and  $K$  be subgroups of a group  $G$ . For  $g \in G$ , define the double coset

$$HgK = \{h g k \mid h \in H, k \in K\}.$$

Then, if  $H$  is normal, we have  $HgH = gH$  for all  $g \in G$ .

b. Let  $GL(n; \mathbb{C})$  be the group of all  $n \times n$  invertible matrices with complex entries. The set of all  $n \times n$  invertible upper triangular matrices is a normal subgroup.

c. Let  $\mathbb{M}(n; \mathbb{R})$  denote the set of all  $n \times n$  matrices with real entries (identified with  $\mathbb{R}^{n^2}$  and endowed with its usual topology) and let  $GL(n; \mathbb{R})$  denote the group of invertible matrices. Let  $G$  be a subgroup of  $GL(n; \mathbb{R})$ . Define

$$H = \left\{ A \in G \mid \begin{array}{l} \text{there exists } \varphi : [0, 1] \rightarrow G \text{ continuous,} \\ \text{such that } \varphi(0) = A, \varphi(1) = I \end{array} \right\}.$$

Then,  $H$  is a normal subgroup of  $G$ .

**1.3** How many (non-isomorphic) groups of order 15 are there?

**1.4** Pick out the true statements:

a. Let  $R$  be a commutative ring with identity. Let  $M$  be an ideal such that every element of  $R$  not in  $M$  is a unit. Then  $R/M$  is a field.

b. Let  $R$  be as above and let  $M$  be an ideal such that  $R/M$  is an integral domain. Then  $M$  is a prime ideal.

c. Let  $R = \mathcal{C}[0, 1]$  be the ring of real-valued continuous functions on  $[0, 1]$  with respect to pointwise addition and pointwise multiplication. Let

$$M = \{f \in R \mid f(0) = f(1) = 0\}.$$

Then  $M$  is a maximal ideal.

**1.5** Write down all the possible values for the degree of an irreducible polynomial in  $\mathbb{R}[x]$ .

**1.6** Let  $V$  be the real vector space of all polynomials in  $\mathbb{R}[x]$  with degree less than, or equal to 4. Consider the linear transformation which maps  $p \in V$  to its derivative  $p'$ . If the matrix of this transformation with respect to the basis  $\{1, x, x^2, x^3, x^4\}$  is  $A$ , write down the matrix  $A^3$ .

**1.7** Let  $\mathbb{T}(n; \mathbb{R}) \subset \mathbb{M}(n; \mathbb{R})$  denote the set of all matrices whose trace is zero. Write down a basis for  $\mathbb{T}(2; \mathbb{R})$ .

**1.8** What is the quotient space  $\mathbb{M}(n; \mathbb{R})/\mathbb{T}(n; \mathbb{R})$  isomorphic to?

**1.9** Construct a  $2 \times 2$  matrix  $A (\neq I)$  with real entries such that  $A^3 = I$ .

**1.10** If  $A \in \mathbb{M}(n; \mathbb{R})$ , let  ${}^tA$  denote its transpose. A matrix  $S \in \mathbb{M}(n; \mathbb{R})$  is said to be *skew-symmetric* if  ${}^tS = -S$ . Pick out the true statements:

- a. If  $S \in \mathbb{M}(n; \mathbb{R})$  is skew-symmetric and non-singular, then  $n$  is even.
- b. Let

$$G = \{T \in GL(n; \mathbb{R}) \mid {}^tTST = S, \text{ for all skew-symmetric } S \in \mathbb{M}(n; \mathbb{R})\}.$$

Then  $G$  is a subgroup of  $GL(n; \mathbb{R})$ .

c. Let  $I_n$  and  $O_n$  denote the  $n \times n$  identity and null matrices respectively. let  $S$  be the  $2n \times 2n$  matrix given in block form by

$$\begin{bmatrix} O_n & I_n \\ -I_n & O_n \end{bmatrix}.$$

If  $X$  is a  $2n \times 2n$  matrix such that  ${}^tXS + SX = 0$ , then the trace of  $X$  is zero.

## Section 2: Analysis

**2.1** Let  $\{a_n\}$  be a sequence of positive terms. Pick out the cases which imply that  $\sum a_n$  is convergent.

a.

$$\lim_{n \rightarrow \infty} n^{\frac{3}{2}} a_n = \frac{3}{2}.$$

b.

$$\sum n^2 a_n^2 < \infty.$$

c.

$$\frac{a_{n+1}}{a_n} < \left( \frac{n}{n+1} \right)^2, \text{ for all } n.$$

**2.2** Evaluate:

$$\lim_{n \rightarrow \infty} \left\{ \frac{1}{1+n^3} + \frac{4}{8+n^3} + \cdots + \frac{n^2}{n^3+n^3} \right\}.$$

**2.3** Find the points in  $\mathbb{R}$  where the following function is differentiable:

$$f(x) = \begin{cases} \tan^{-1} x, & \text{if } |x| \leq 1 \\ \frac{\pi}{4} \operatorname{sgn}(x) + \frac{|x|-1}{2}, & \text{if } |x| > 1, \end{cases}$$

where  $\operatorname{sgn}(x)$  equals  $+1$  if  $x > 0$ ,  $-1$  if  $x < 0$  and is equal to zero if  $x = 0$  and  $\tan^{-1}(x)$  takes its values in the range  $]-\pi/2, \pi/2[$  for real numbers  $x$ .

**2.4** Pick out the true statements:

a. If  $P$  is a polynomial in one variable with real coefficients which has all its roots real, then its derivative  $P'$  has all its roots real as well.

b. The equation  $\cos(\sin x) = x$  has exactly one solution in the interval  $[0, \frac{\pi}{2}]$ .

c.  $\cos x > 1 - \frac{x^2}{2}$  for all  $x > 0$ .

**2.5** Let  $f, f_n : [0, 1] \rightarrow \mathbb{R}$  be continuous functions. Complete the following sentence such that both statements (a) and (b) below are true:

“Let  $f_n \rightarrow f$  .....”

a.

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = \int_0^1 f(x) dx.$$

b.

$$\lim_{n \rightarrow \infty} \lim_{x \rightarrow 0} f_n(x) = \lim_{x \rightarrow 0} \lim_{n \rightarrow \infty} f_n(x).$$

**2.6** Let  $f : ]0, 1[ \rightarrow \mathbb{R}$  be continuous. Pick out the statements which imply that  $f$  is uniformly continuous.

a.  $|f(x) - f(y)| \leq \sqrt{|x - y|}$ , for all  $x, y \in ]0, 1[$ .

b.  $f(1/n) \rightarrow 1/2$  and  $f(1/n^2) \rightarrow 1/4$ .

c.

$$f(x) = x^{\frac{1}{2}} \sin \frac{1}{x^3}.$$

**2.7** Evaluate:

$$\int \int_{[0,1] \times [0,1]} \max\{x, y\} \, dx dy.$$

**2.8** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be analytic. Pick out the cases when  $f$  is **not** necessarily a constant.

- a.  $\operatorname{Im}(f'(z)) > 0$  for all  $z \in \mathbb{C}$ .
- b.  $f(n) = 3$  for all  $n \in \mathbb{Z}$ .
- c.  $f'(0) = 0$  and  $|f'(z)| \leq 3$  for all  $z \in \mathbb{C}$ .

**2.9** Let  $f : \mathbb{C} \rightarrow \mathbb{C}$  be analytic. Write  $z = x + iy$  and  $f = u + iv$ , where  $u$  and  $v$  are real valued functions of  $x$  and  $y$ . Pick out the true statements.

a.

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

b.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

c.

$$f''(0) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z^3} \, dz.$$

**2.10** Find the square roots of  $1 + i\sqrt{3}$ .

### Section 3: Topology

**3.1** Which of the following define a metric?

- $d((x, y), (x', y')) = \min\{|x - x'|, |y - y'|\}$  on  $\mathbb{R}^2$ .
- $d((x, y), (x', y')) = |x| + |y| + |x'| + |y'|$  on  $\mathbb{R}^2$ .
- $D((x, y), (x', y')) = d(x, x') + d(y, y')$  on  $X \times X$ , where  $(X, d)$  is a metric space.

**3.2** Let  $(X, d)$  be a metric space and let  $A \subset X$ . For  $x \in X$  define

$$d(x, A) = \inf\{d(x, y) \mid y \in A\}.$$

Pick out the true statements:

- $x \mapsto d(x, A)$  is a uniformly continuous function.
- If

$$\partial A = \{x \in X \mid d(x, A) = 0\} \cap \{x \in X \mid d(x, X \setminus A) = 0\},$$

then  $\partial A$  is closed for any  $A \subset X$ .

- Let  $A$  and  $B$  be subsets of  $X$  and define

$$d(A, B) = \inf\{d(a, B) \mid a \in A\}.$$

Then  $d(A, B) = d(B, A)$ .

**3.3** Let  $X$  be a topological space and for  $A \subset X$ , denote by  $\overline{A}$  and  $A^\circ$ , the closure and interior of  $A$  respectively. Pick out the true statements.

- $\overline{A \cup B} = \overline{A} \cup \overline{B}$ .
- $\overline{A \cap B} = \overline{A} \cap \overline{B}$ .
- Consider  $\mathbb{R}$  as the  $x$ -axis in  $\mathbb{R}^2$ . Then  $\mathbb{R}^\circ = \emptyset$ .

**3.4** Pick out the true statements.

- Let  $\{X_i\}_{i \in \mathcal{I}}$  be topological spaces. Then, the product topology is the smallest topology on  $X = \prod_{i \in \mathcal{I}} X_i$  such that each of the canonical projections  $p_i : X \rightarrow X_i$  is continuous.
- Let  $X$  be a topological space and  $W \subset X$ . Then, the induced subspace topology on  $W$  is the smallest topology such that  $id|_W : W \rightarrow X$ , where  $id$  is the identity map, is continuous.
- Let  $X = \mathbb{R}^n$  with the usual topology. This is the smallest topology such that all linear functionals on  $X$  are continuous.

**3.5** Which of the following subsets are dense in the given spaces?

- The set of trigonometric polynomials in the space of continuous functions on  $[-\pi, \pi]$  which are  $2\pi$ -periodic (with the sup-norm topology).
- The subset of  $C^\infty$  functions with compact support in  $\mathbb{R}$  in the space of bounded real-valued continuous functions on  $\mathbb{R}$  (with the sup-norm topology).
- $GL(n; \mathbb{R})$  in  $M(n; \mathbb{R})$  (with its usual topology after identification with  $\mathbb{R}^{n^2}$ ).

**3.6** Pick out the compact sets.

- $\{(x, y) \mid x^2 - y^2 = 1\} \subset \mathbb{R}^2$ .
- $\{\text{Tr}(A) \mid A \in M(n; \mathbb{R}), A \text{ orthogonal}\} \subset \mathbb{R}$ , where  $\text{Tr}(A)$  denotes the trace of the matrix  $A$ .
- The set of all matrices in  $M(n; \mathbb{R})$  all of whose eigenvalues satisfy the condition  $|\lambda| \leq 2$ .

**3.7** Pick out the connected sets.

- $\{(x, y) \mid xy = 1\} \subset \mathbb{R}^2$ .
- The set of all upper triangular matrices in  $M(n; \mathbb{R})$ .
- The set of all invertible diagonal matrices in  $M(n; \mathbb{R})$ .

**3.8** Pick out the true statements.

- Let  $f : \mathbb{Z} \rightarrow \mathbb{Z}^2$  be a bijection. There exists a continuous function from  $\mathbb{R}$  to  $\mathbb{R}^2$  which extends  $f$ .
- Let  $D$  denote the closed unit disc in  $\mathbb{R}^2$ . There exists a continuous mapping  $f : D \setminus \{(0, 0)\} \rightarrow \{x \in \mathbb{R} \mid |x| \leq 1\}$  which is onto.
- Let  $D$  denote the closed unit disc in  $\mathbb{R}^2$ . There exists a continuous mapping  $f : D \setminus \{(0, 0)\} \rightarrow \{x \in \mathbb{R} \mid |x| > 1\}$  which is onto.

**3.9** A Hausdorff topological space is said to be normal if given any two disjoint closed sets  $A$  and  $B$ , there exist disjoint open sets  $U$  and  $V$  such that  $A \subset U$  and  $B \subset V$ . Pick out the true statements.

- Every metric space is normal.
- If  $X$  is a normal space with at least two distinct points, then there exist non-constant real-valued continuous functions on  $X$ .
- If  $X$  is normal and  $Y \subset X$  is closed, then  $Y$  is normal for the induced topology.

**3.10** Which of the following pairs of sets are homeomorphic?

- $A = \{(x, y) \mid x^2 + y^2 - 2x + 4y - 5 = 0\}$  and  $B = \{(x, y) \mid 5x^2 + 3y^2 = 1\}$ .
- $A = \{(x, y) \mid x^2 + y^2 - 2x + 4y - 5 = 0\}$  and  $B = \{(x, y) \mid 5x^2 - 3y^2 = 1\}$ .
- $A = \{(x, y) \mid x^2 + y^2 - 2x + 4y - 5 \leq 0\}$  and  $B = \{(x, y) \mid 5x^2 + 3y^2 \geq 1\}$ .



## Section 4: Applied Mathematics

**4.1** Simpson's rule is used to approximate the integral  $\int_0^1 f(x) dx$ . If  $f$  is a polynomial, what is the maximum possible degree it can have so that Simpson's rule gives the exact value of this integral?

**4.2** A right circular cylinder of fixed volume has maximum total surface area. What is the relationship between its height  $h$  and radius  $r$ ?

**4.3** In the equations governing the flow of an incompressible fluid of uniform density, if  $\mathbf{u}$  is the velocity vector and  $p$  is the pressure, write down the equation which expresses the law of conservation of mass.

**4.4** A particle of mass  $M$  is attached to a fixed wall by a spring. The spring exerts no force when the particle is at its equilibrium position at  $x = 0$  and exerts a restoring force proportional to the displacement when it is displaced to a distance  $x$ . In addition, there is a damping force due to the medium in which the displacement takes place, which is a force opposing the motion and is proportional to the velocity of the particle. If the particle is pulled to a position  $x_0$  at time  $t = 0$  and is released without any velocity, write down the initial value problem governing the motion of the particle.

**4.5** Solve the following linear programming problem:

$$\begin{aligned}\max z &= 5x + 7y \\ x - y &\leq 1 \\ 2x + y &\geq 2 \\ x + 2y &\leq 4 \\ x, y &\geq 0.\end{aligned}$$

**4.6** Write down the dual of the above problem.

**4.7** Find the general solution of the system:

$$\begin{aligned}\frac{dx}{dt} &= x + y \\ \frac{dy}{dt} &= 4x - 2y.\end{aligned}$$

**4.8** Let  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  be vectors in  $\mathbb{R}^3$ . Express  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$  as a linear combination of  $\mathbf{b}$  and  $\mathbf{c}$ .

**4.9** Solve:

$$\frac{\partial^2 u}{\partial x^2} = 6xy; \quad u(0, y) = y; \quad \frac{\partial u}{\partial x}(1, y) = 0.$$

**4.10** Let  $\Omega$  be a smooth plane domain of unit area. Let  $u(x, y) = 3x^2 + y^2$ . If  $\frac{\partial u}{\partial n}$  denotes its outer normal derivative on  $\partial\Omega$ , the boundary of  $\Omega$ , compute

$$\int_{\partial\Omega} \frac{\partial u}{\partial n} ds.$$

## Section 5: Miscellaneous

**5.1** Let  $V$  be a real vector space of real-valued functions on a given set. Assume that constant functions are in  $V$  and that if  $f \in V$ , then  $f^2 \in V$  and that  $|f| \in V$ . Pick out the true statements.

- If  $f, g \in V$ , then  $fg \in V$ .
- If  $f, g \in V$ , then  $\max\{f, g\} \in V$ .
- If  $f \in V$  and  $p$  is any polynomial in one variable, with real coefficients, then  $p(f) \in V$ .

**5.2** A fair coin is tossed 10 times, the tosses being independent of each other. Find the probability that the results of the third, fourth and fifth tosses are identical.

**5.3** Determine if the following collections are countable or uncountable.

- The collection of all finite subsets of  $\mathbb{N}$ .
- The collection of all infinite sequences of positive integers.
- The collection of all roots of all polynomials in one variable, with integer coefficients.

**5.4** Find the maximum value of  $x + 2y + 3z$  subject to the constraint  $x^2 + y^2 + z^2 = 1$ .

**5.5** Let  $A_n$  be the  $n \times n$  matrix whose  $(i, j)$ -th entry is given by

$$2\delta_{ij} - \delta_{i+1,j} - \delta_{i,j+1}$$

where  $\delta_{ij}$  equals 1 if  $i = j$  and zero otherwise. Compute the determinant of  $A_n$ .

**5.6** How many real roots does the following equation have?

$$3^x + 4^x = 5^x$$

**5.7** Let  $N > 1$  be a positive integer. Let  $\phi(N)$  denote the number of positive integers less than  $N$  and prime to it (unity being included in this count). Express the sum of all the integers less than  $N$  and prime to it in terms of  $\phi(N)$ .

**5.8** Pick out the true statements.

- The sum of  $r$  consecutive positive integers is divisible by  $r$ .
- The product of  $r$  consecutive positive integers is divisible by  $r!$ .
- For each positive integer  $r$ , there exist  $r$  consecutive positive integers which are all composite.

**5.9** Let  $n$  be a fixed positive integer and let  $0 \leq k \leq n$ . We denote by  $C_k$ , the number of ways of choosing  $k$  objects from  $n$  distinct objects. Sum to  $n$  terms:

$$3C_1 + 7C_2 + 11C_3 + \cdots$$

**5.10** Find the sum of the following infinite series:

$$\frac{1}{5} - \frac{1.4}{5.10} + \frac{1.4.7}{5.10.15} - \cdots$$