Blowup-polynomials and delta-matriods of graphs

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Distance matrices of graphs

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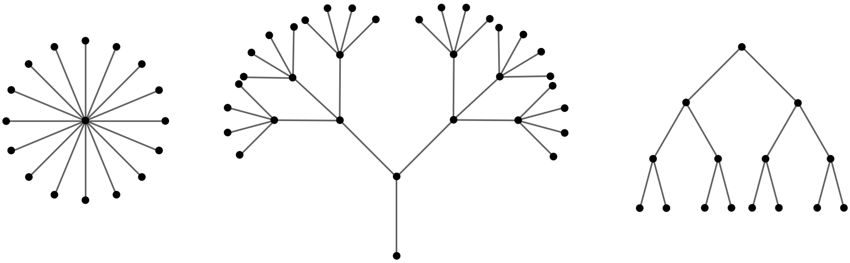
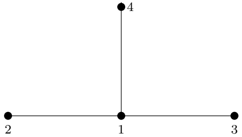
- The *distance matrix* D_G is a $V \times V$ matrix with entries $d(v, w)$.
- Extensively studied quantity: the determinant of D_G for G a tree.



Algebraic fact: The Graham–Pollak result

Examples of distance matrices (on 4 nodes):

T_1, T_2 are the star graph $K_{1,3}$ and the path graph P_4, respectively.



$$D_{T_1} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 \\ 1 & 2 & 0 & 2 \\ 1 & 2 & 2 & 0 \end{pmatrix} \qquad D_{T_2} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

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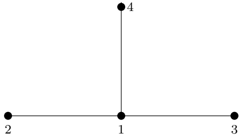

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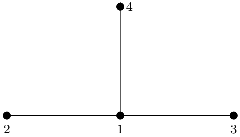

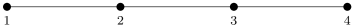

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Theorem (Graham–Pollak, *Bell Sys. Tech. J.*, 1971)

Given a tree T on n nodes, $\det D_T = (-1)^{n-1} 2^{n-2} (n-1)$.

Analysis fact: co-spectral matrices

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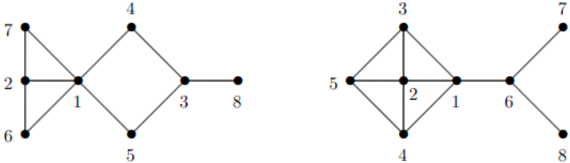

Thus, $\det(D_G - x\,\mathrm{Id}_V)$ does not detect the graph (up to isomorphism).

Inter-related Motivations/Goals:

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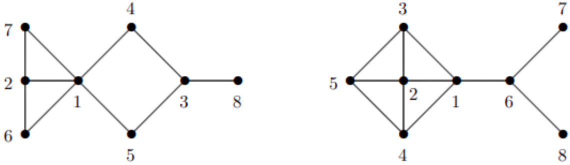

Thus, $\det(D_G - x\,\mathrm{Id}_V)$ does not detect the graph (up to isomorphism).

Inter-related Motivations/Goals:

1. Find a(nother) family $\{G_i : i \in I\}$ of graphs (e.g., trees on k vertices) such that $i \mapsto \det D_{G_i}$ is a "nice" function.
2. Find an invariant of the matrix D_G which detects G (and is related to the distance spectrum – eigenvalues of D_G).

Graph blowups

The key construction is of a *graph blowup* $G[\mathbf{n}]$, where $\mathbf{n} = (n_v)_{v \in V}$ is a V-tuple of positive integers. This is a finite simple connected graph $G[\mathbf{n}]$, with:

- n_v copies of the vertex $v \in V$, and
- a copy of vertex v and one of w are adjacent in $G[\mathbf{n}]$ if and only if $v \neq w$ and v, w are adjacent in G.

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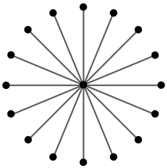
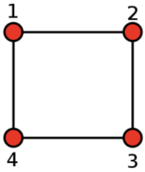


Star graph: $K_{1,n} \cong K_2[(1,n)]$

4-cycle: $C_4 \cong K_2[(2,2)]$.



Distance matrix of graph blowup, and its determinant

Suggestive example: Compute $\det D_{G[\mathbf{n}]}$ in all examples above:

$$\det D_{K_2[(r,s)]} = (-2)^{r+s-2}(3rs - 4r - 4s + 4).$$

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There exists a real polynomial $p_G(\mathbf{n})$ in the sizes n_v, such that:

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Moreover, p_G is multi-affine in $\mathbf{n}$, with constant term $(-2)^{|V|}$ and linear term $-(-2)^{|V|} \sum_{v \in V} n_v$. (In fact, have closed-form expression for every monomial.)

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Definition: Define $p_G(\cdot)$ to be the *blowup-polynomial* of G.
This achieves **Goal 1:** the function $\mathbf{n} \mapsto \det D_{G[\mathbf{n}]}$ is a "nice" function of $\mathbf{n}$, for all graphs G.

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What about **Goal 2** – can p_G recover G?

p_G is a graph invariant

Note: If G has an automorphism sending a vertex $v \in V$ to w, then the blowup-polynomial is "symmetric" under $n_v \longleftrightarrow n_w$.

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(Answers **Goal 2**.)

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- The polynomial $u_{K_2}(n) = 3n^2 - 8n + 4 = (n-2)(3n-2)$.
- More generally: $u_{K_k}(n) = (n-2)^{k-1}(kn+n-2)$ – also real rooted.

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Taken forward by Marcus–Spielman–Srivastava:

- Proved the Kadison–Singer conjecture. [*Ann. of Math.* 2015]
- Existence of bipartite Ramanujan graphs of all degrees and orders – proved conjectures of Bilu–Linial and Lubotzky. [*Ann. of Math.* 2015]

Beyond real-stability: Lorenztian / strongly Rayleigh

Recall from above (with $|V| = k$) that $p_G(\mathbf{z})$ has constant term $(-2)^k$ and linear term $-(-2)^k \sum_{j=1}^{k} z_j$.

Thus, the real-stable polynomial p_G does not satisfy two further properties:

1. The coefficients are not all of the same sign. [Can consider $p_G(-\mathbf{z})$.]
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Our next result characterizes the graphs for which this holds:

Strongly Rayleigh graphs are complete multi-partite

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Given a graph G as above, define its homogenized blowup-polynomial

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3. $(-1)^k p_G(-1, \dots, -1) > 0$, *and the normalized "reflected" polynomial*
$$q_G : (z_1, \dots, z_k) \quad \mapsto \quad \frac{p_G(-z_1, \dots, -z_k)}{p_G(-1, \dots, -1)}$$
is strongly Rayleigh, *i.e.,* q_G *is real-stable, has non-negative coefficients (of all monomials $\prod_{j \in J} z_j$), and these sum up to 1.*

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4. *G is a complete multipartite graph.*

Novel characterization of a class of graphs, via real-stability.

Matroids

A *matroid* is a notion common to linear algebra and graph theory (among other areas):

1. A finite set E (called the *ground set*);
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1. *Free matroid:* All subsets of E.
2. *Uniform matroid:* All subsets of E of size $\leq k$ (for fixed k).
3. $E =$ finite subset of vector space; $\mathcal{F} =$ linearly independent subsets of E. (E.g., *Linear matroid*: E indexes the columns of a matrix A over a field.)

Delta-matroids

A related well-studied notion is that of a *delta-matroid*.

Example 1: Restrict to the *bases* of $\mathrm{Col}(A)$, not all linearly independent subsets. These satisfy the "Symmetric Exchange Axiom":

$$A, B \in \mathcal{F},\ x \in A \Delta B \quad \Longrightarrow \quad \text{there exists } y \in A\Delta B \ \text{ s.t. } A\Delta\{x, y\} \in \mathcal{F}.$$

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In general, a delta-matroid consists of:

1. A finite *ground set* E;
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Example 2: *Linear delta-matroid* – given a symmetric or skew-symmetric matrix $A_{n\times n}$ over a field, let $E := \{1, \dots, n\}$.

A subset $F \subseteq E$ is *feasible* $\iff$ $\det A_{F\times F} \neq 0$.
The set of feasible subsets is the linear delta-matroid, denoted by $\mathcal{M}_A$.

From blowup-polynomials to blowup delta-matroids

Brändén (*Adv. Math.* 2007) showed: if $p(z_1, \dots, z_k)$ is a real-stable multi-affine polynomial, then the set of monomials in p forms a delta-matroid with ground set $E = \{1, \dots, k\}$.

Thus, every blowup-polynomial $p_G(\cdot)$ is real-stable $\rightsquigarrow$ (novel) delta-matroid.

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In fact, this delta-matroid is linear: $\mathcal{M}_{M_G}$.

Example: For $G = P_3$ (path graph), with $E = \{1, 2, 3\}$,

$$\mathcal{M}_{M_{P_3}} = 2^E \setminus \{\{1, 3\}, \{1, 2, 3\}\}.$$

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Questions:

1. Does this hold for all k?
2. Regardless of (1), is the right-hand side a delta-matroid for all k?

Another delta-matroid for trees

Proposition (C.–Khare, 2021)

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In particular, for $k \geq 9$, the right-hand side yields a different novel delta-matroid for P_k. How to generalize this phenomenon?

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- The induced subgraph in P_k on $I := \{i, i+1, i+2\}$ is a tree which is a blowup-graph: $P_3 = K_2[(2,1)]$, and $i, i+2$ are copies of a vertex in K_2.
- More generally, any tree containing two leaves with common parent, is a blowup. Declare all such subsets of nodes to be infeasible. Does this yield a delta-matroid?

Another delta-matroid for trees

Theorem (C.–Khare, 2021)

Suppose T is a tree. Define a subset of vertices I to be infeasible if its Steiner tree $T(I)$ has two leaves, which are in I and have the same parent. Then the remaining, "feasible" subsets form a delta-matroid $\mathcal{M}'(T)$.

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- Novel delta-matroid arising from *combinatorics*.
- We also show that the construction of $\mathcal{M}'(T)$ does *not* extend to arbitrary graphs.
- Connection to other known, combinatorial delta-matroids?

References

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