Approximation of weak G-bundles in high dimensions

Swarnendu Sil

Forschungsinstitut für Mathematik
ETH Zürich

9th March 2020
Geometry and Topology Seminar
Indian Institute of Science
Bengaluru, India

1. **Gauge theory on Riemannian manifolds**
   - Bundles, connections and curvatures
   - Yang-Mills Functional
   - Main questions in higher dimensions
   - Weak compactness, Coulomb gauges and topology change
2. **Approximation for manifold-valued Sobolev maps**
   - Topology below the continuity threshold
   - Approximation of manifold-valued Sobolev maps
3. **Approximation and topology for bundles with connections**
   - Approximation for vanishing Morrey-Sobolev bundles with connections
   - Regularity of Coulomb bundles
   - Topology for bundle-connection pair
4. **The road ahead**
   - What's next?: Approximation by 'almost' smooth classes
   - Concluding words
5. **Bonus slide: Instantons**

Gauge theory on Riemannian manifolds

Basic objects in gauge theory

Gauge theory is the study of the critical points of the **Yang-Mills** functional

$$YM\left(A\right) := \int_{M^n} \left|F_A\right|^2 := -\int_{M^n} \operatorname{Tr}\left(F_A \wedge *F_A\right).$$

- M^n is n-dim smooth compact oriented Riemannian manifold (mostly closed)
- G is a finite dim compact **Lie group**: typically $\mathrm{O}(m), \mathrm{SO}(m)$, $\mathrm{U}(m)$, $\mathrm{SU}(m)$
- A is a **principal connection** on a **principal** G**-bundle** over M^n
- F_A is the **curvature** of the connection A.

Gauge theory on Riemannian manifolds

Basic objects in gauge theory

Gauge theory is the study of the critical points of the **Yang-Mills** functional

$$YM(A) := \int_{M^n} |F_A|^2 := -\int_{M^n} \mathrm{Tr}\,(F_A \wedge *F_A)\,.$$

- M^n is n-dim smooth compact oriented Riemannian manifold (mostly closed)
- G is a finite dim compact **Lie group**: typically $\mathrm{O}(m), \mathrm{SO}(m)$, $\mathrm{U}(m)$, $\mathrm{SU}(m)$
- A is a **principal connection** on a **principal** G-**bundle** over M^n
- F_A is the **curvature** of the connection A.

$\mathrm{SU}(2)$ - unit quaternions, $\mathfrak{su}(2)$ - traceless 2×2 cx. skew-Hermitian matrices.

Imp. Principal $\mathrm{SU}(2)$-bundles: frame bundles for Quaternion line bundles

$c_2 = -\frac{1}{2}p_1 = \chi$ classifies principal $\mathrm{SU}(2)$ bundles (and $\mathbb{H}$ line bundles) over M^4.
The Hopf fibration $\pi : \mathbb{S}^7 \to \mathbb{S}^4 \sim$ the tautological line bundle over $\mathbb{HP}^1$.

Principal G-bundles and connections

Smooth and C^0 Principal G-bundles

A **smooth principal G-bundle** P over M^n, denoted $\pi : P \to M^n$, is locally just a product, i.e. for $M^n = \bigcup_{\alpha \in I} U_\alpha$, we have $P|_{U_\alpha} \simeq U_\alpha \times G$.

Principal G-bundles and connections

Smooth and C^0 Principal G-bundles

A **smooth principal G-bundle** P over M^n, denoted $\pi : P \to M^n$, is locally just a product, i.e. for $M^n = \bigcup_{\alpha \in I} U_\alpha$, we have $P|_{U_\alpha} \simeq U_\alpha \times G$.

Bundle trivialization maps and transition maps

Bundle trivialization maps: For every $\alpha \in I$, fiber-preserving smooth diffeos

$$\phi_\alpha : U_\alpha \times G \to \pi^{-1}(U_\alpha),$$

which are G-equivariant:

Principal G-bundles and connections

Smooth and C^0 Principal G-bundles

A **smooth principal G-bundle** P over M^n, denoted $\pi : P \to M^n$, is locally just a product, i.e. for $M^n = \bigcup_{\alpha \in I} U_\alpha$, we have $P|_{U_\alpha} \simeq U_\alpha \times G$.

Bundle trivialization maps and transition maps

Bundle trivialization maps: For every $\alpha \in I$, fiber-preserving smooth diffeos

$$\phi_\alpha : U_\alpha \times G \to \pi^{-1}(U_\alpha),$$

which are G-equivariant: i.e. whenever $U_\alpha \cap U_\beta \neq \emptyset$, there exist smooth maps, called **transition functions** (or **clutching functions**)

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$$

such that for every $h \in G$ and every $x \in U_\alpha \cap U_\beta$, we have

$$\left(\phi_\alpha^{-1} \circ \phi_\beta\right)(x, h) = (x, g_{\alpha\beta}(x)h). \tag{1}$$

Bundles as transition function data

Cocycle conditions

From the relation (1), the transition functions satisfy the **cocycle identity**

$$g_{\alpha\beta}(x)g_{\beta\gamma}(x) = g_{\alpha\gamma}(x) \qquad \text{for every } x \in U_\alpha \cap U_\beta \cap U_\gamma. \tag{2}$$

Bundles as transition function data

Cocycle conditions

From the relation (1), the transition functions satisfy the **cocycle identity**

$$g_{\alpha\beta}(x)g_{\beta\gamma}(x) = g_{\alpha\gamma}(x) \qquad \text{for every } x \in U_\alpha \cap U_\beta \cap U_\gamma. \tag{2}$$

Bundles as transition function data

$$P = \left(\{U_\alpha\}_{\alpha \in I}, \{g_{\alpha\beta}\}_{\alpha,\beta \in I} \right) \in \mathcal{P}_G^\infty(M^n), \text{ or } \mathcal{P}_G^0(M^n) \text{ if } g_{\alpha\beta} \text{ only } C^0.$$

Bundles as transition function data

Cocycle conditions

From the relation (1), the transition functions satisfy the **cocycle identity**

$$g_{\alpha\beta}(x)g_{\beta\gamma}(x) = g_{\alpha\gamma}(x) \qquad \text{for every } x \in U_\alpha \cap U_\beta \cap U_\gamma. \tag{2}$$

Bundles as transition function data

$$P = \left(\{U_\alpha\}_{\alpha \in I}, \{g_{\alpha\beta}\}_{\alpha,\beta \in I} \right) \in \mathcal{P}^\infty_G(M^n), \text{ or } \mathcal{P}^0_G(M^n) \text{ if } g_{\alpha\beta} \text{ only } C^0.$$

Two C^0 bundles $P, Q \in \mathcal{P}^0_G(M^n)$ are **C^0-equivalent** , denoted by $[P]_{C^0} = [Q]_{C^0}$, if there are **continuous** maps $\sigma_\alpha : U_\alpha \to G$ such that

$$h_{\alpha\beta} = \sigma_\alpha^{-1} g_{\alpha\beta} \sigma_\beta \quad \text{in } U_\alpha \cap U_\beta.$$

C^0-equivalent bundles are the 'same' for topology.

Bundles as transition function data

Cocycle conditions

From the relation (1), the transition functions satisfy the **cocycle identity**

$$g_{\alpha\beta}(x)g_{\beta\gamma}(x) = g_{\alpha\gamma}(x) \qquad \text{for every } x \in U_\alpha \cap U_\beta \cap U_\gamma. \tag{2}$$

Bundles as transition function data

$$P = \left(\{U_\alpha\}_{\alpha\in I}, \{g_{\alpha\beta}\}_{\alpha,\beta\in I} \right) \in \mathcal{P}^\infty_G(M^n), \text{ or } \mathcal{P}^0_G(M^n) \text{ if } g_{\alpha\beta} \text{ only } C^0.$$

Two C^0 bundles $P, Q \in \mathcal{P}^0_G(M^n)$ are **C^0-equivalent** , denoted by $[P]_{C^0} = [Q]_{C^0}$, if there are **continuous** maps $\sigma_\alpha : U_\alpha \to G$ such that

$$h_{\alpha\beta} = \sigma_\alpha^{-1} g_{\alpha\beta} \sigma_\beta \quad \text{ in } U_\alpha \cap U_\beta.$$

C^0-equivalent bundles are the 'same' for topology. This is a sheaf-theoretic description and bundles are basically a Čech cohomology class (non-Abelian!).

Connection and gauges

Connection (Ehresmann/Principal) on a smooth principal G-bundle

Locally, $A \in \mathcal{A}^{\infty}(P)$ is given by smooth $A_{\alpha} : U_{\alpha} \to \Lambda^{1} T^{*} U_{\alpha} \otimes \mathfrak{g}$ satisfying the **gluing relations**

$$A_{\beta} = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} A_{\alpha} g_{\alpha\beta} \qquad \text{in } U_{\alpha} \cap U_{\beta}. \tag{3}$$

Connection and gauges

Connection (Ehresmann/Principal) on a smooth principal G-bundle

Locally, $A \in \mathcal{A}^{\infty}(P)$ is given by smooth $A_{\alpha} : U_{\alpha} \to \Lambda^{1} T^{*} U_{\alpha} \otimes \mathfrak{g}$ satisfying the **gluing relations**

$$A_{\beta} = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} A_{\alpha} g_{\alpha\beta} \qquad \text{in } U_{\alpha} \cap U_{\beta}. \tag{3}$$

Gauges: Maps $\rho_{\alpha} : U_{\alpha} \to G$.

Connection and gauges

Connection (Ehresmann/Principal) on a smooth principal G-bundle

Locally, $A \in \mathcal{A}^{\infty}(P)$ is given by smooth $A_{\alpha} : U_{\alpha} \to \Lambda^1 T^* U_{\alpha} \otimes \mathfrak{g}$ satisfying the **gluing relations**

$$A_{\beta} = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} A_{\alpha} g_{\alpha\beta} \qquad \text{in } U_{\alpha} \cap U_{\beta}. \tag{3}$$

Gauges: Maps $\rho_{\alpha} : U_{\alpha} \to G$. Induces a change of trivializations

$$\phi_{\alpha}^{\rho_{\alpha}}(x, h) = \phi_{\alpha}(x, \rho_{\alpha}(x)h) \qquad \text{for all } x \in U_{\alpha} \text{ and for all } h \in G.$$

Connection and gauges

Connection (Ehresmann/Principal) on a smooth principal G-bundle

Locally, $A \in \mathcal{A}^{\infty}(P)$ is given by smooth $A_{\alpha} : U_{\alpha} \to \Lambda^1 T^* U_{\alpha} \otimes \mathfrak{g}$ satisfying the **gluing relations**

$$A_{\beta} = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} A_{\alpha} g_{\alpha\beta} \qquad \text{in } U_{\alpha} \cap U_{\beta}. \tag{3}$$

Gauges: Maps $\rho_{\alpha} : U_{\alpha} \to G$. Induces a change of trivializations

$$\phi_{\alpha}^{\rho_{\alpha}}(x, h) = \phi_{\alpha}(x, \rho_{\alpha}(x)h) \qquad \text{for all } x \in U_{\alpha} \text{ and for all } h \in G.$$

$\{A_{\alpha}^{\rho_{\alpha}}\}_{\alpha \in I}$ satisfy the **gauge change identities**

$$A_{\alpha}^{\rho_{\alpha}} = \rho_{\alpha}^{-1} d\rho_{\alpha} + \rho_{\alpha}^{-1} A_{\alpha} \rho_{\alpha} \qquad \text{in } U_{\alpha}. \tag{4}$$

Connection and gauges

Connection (Ehresmann/Principal) on a smooth principal G-bundle

Locally, $A \in \mathcal{A}^{\infty}(P)$ is given by smooth $A_{\alpha} : U_{\alpha} \to \Lambda^{1} T^{*} U_{\alpha} \otimes \mathfrak{g}$ satisfying the **gluing relations**

$$A_{\beta} = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} A_{\alpha} g_{\alpha\beta} \qquad \text{in } U_{\alpha} \cap U_{\beta}. \tag{3}$$

Gauges: Maps $\rho_{\alpha} : U_{\alpha} \to G$. Induces a change of trivializations

$$\phi_{\alpha}^{\rho_{\alpha}}(x, h) = \phi_{\alpha}(x, \rho_{\alpha}(x)h) \qquad \text{for all } x \in U_{\alpha} \text{ and for all } h \in G.$$

$\{A_{\alpha}^{\rho_{\alpha}}\}_{\alpha \in I}$ satisfy the **gauge change identities**

$$A_{\alpha}^{\rho_{\alpha}} = \rho_{\alpha}^{-1} d\rho_{\alpha} + \rho_{\alpha}^{-1} A_{\alpha} \rho_{\alpha} \qquad \text{in } U_{\alpha}. \tag{4}$$

New transition functions:

$$h_{\alpha\beta} = \rho_{\alpha}^{-1} g_{\alpha\beta} \rho_{\beta} \qquad \text{in } U_{\alpha} \cap U_{\beta}.$$

Curvature and Yang-Mills energy

Curvature of a connection

$F_{A_\alpha} : U_\alpha \to \Lambda^2 T^* U_\alpha \otimes \mathfrak{g}$ are the local expressions of the *curvature* of A, given by

$$F_{A_\alpha} = dA_\alpha + A_\alpha \wedge A_\alpha = dA_\alpha + \frac{1}{2} [A_\alpha, A_\alpha] \qquad \text{in } U_\alpha, \tag{5}$$

where

$$A_\alpha \wedge A_\alpha = \sum_{i,j} A_i A_j dx_i \wedge dx_j = \sum_{i<j} [A_i, A_j] \, dx_i \wedge dx_j.$$

Curvature and Yang-Mills energy

Curvature of a connection

$F_{A_\alpha} : U_\alpha \to \Lambda^2 T^* U_\alpha \otimes \mathfrak{g}$ are the local expressions of the *curvature* of A, given by

$$F_{A_\alpha} = dA_\alpha + A_\alpha \wedge A_\alpha = dA_\alpha + \frac{1}{2}\left[A_\alpha, A_\alpha\right] \qquad \text{in } U_\alpha, \tag{5}$$

where

$$A_\alpha \wedge A_\alpha = \sum_{i,j} A_i A_j dx_i \wedge dx_j = \sum_{i<j} \left[A_i, A_j\right] dx_i \wedge dx_j.$$

gluing relation:

$$F_{A_\beta} = g_{\alpha\beta}^{-1} F_{A_\alpha} g_{\alpha\beta} \qquad \text{in } U_\alpha \cap U_\beta.$$

Curvature and Yang-Mills energy

Curvature of a connection

$F_{A_\alpha} : U_\alpha \to \Lambda^2 T^* U_\alpha \otimes \mathfrak{g}$ are the local expressions of the *curvature* of A, given by

$$F_{A_\alpha} = dA_\alpha + A_\alpha \wedge A_\alpha = dA_\alpha + \frac{1}{2} [A_\alpha, A_\alpha] \qquad \text{in } U_\alpha, \tag{5}$$

where

$$A_\alpha \wedge A_\alpha = \sum_{i,j} A_i A_j dx_i \wedge dx_j = \sum_{i<j} [A_i, A_j] \, dx_i \wedge dx_j .$$

gluing relation:

$$F_{A_\beta} = g_{\alpha\beta}^{-1} F_{A_\alpha} g_{\alpha\beta} \qquad \text{in } U_\alpha \cap U_\beta .$$

gauge change identities:

$$F_{A_\alpha^{\rho_\alpha}} = \rho_\alpha^{-1} F_{A_\alpha} \rho_\alpha \qquad \text{in } U_\alpha . \tag{6}$$

Yang-Mills energy

Yang-Mills energy

$$YM(A) := \int_{M^n} |F_A|^2 := -\int_{M^n} \operatorname{Tr}(F_A \wedge *F_A).$$

The invariance of the *Killing scalar product* together with (6) implies that the norm $|F_A|$ is gauge invariant and so is YM.

Yang-Mills energy

Yang-Mills energy

$$YM(A) := \int_{M^n} |F_A|^2 := -\int_{M^n} \mathrm{Tr}\,(F_A \wedge *F_A)\,.$$

The invariance of the *Killing scalar product* together with (6) implies that the norm $|F_A|$ is gauge invariant and so is YM.

Yang-Mills fields

Euler-Lagrange equation is the (elliptic) Yang-Mills Equation

$d_A^* F_A = 0$ **(YM)**

Yang-Mills energy

Yang-Mills energy

$$YM(A) := \int_{M^n} |F_A|^2 := -\int_{M^n} \mathrm{Tr}\,(F_A \wedge *F_A)\,.$$

The invariance of the *Killing scalar product* together with (6) implies that the norm $|F_A|$ is gauge invariant and so is YM.

Yang-Mills fields

Euler-Lagrange equation is the (elliptic) Yang-Mills Equation

$$d_A^* F_A = 0 \quad \textbf{(YM)} \qquad \text{and} \qquad d_A F_A = 0 \qquad \textbf{(Bianchi identity)}.$$

Yang-Mills energy

Yang-Mills energy

$$YM(A) := \int_{M^n} |F_A|^2 := -\int_{M^n} \mathrm{Tr}\,(F_A \wedge *F_A)\,.$$

The invariance of the *Killing scalar product* together with (6) implies that the norm $|F_A|$ is gauge invariant and so is YM.

Yang-Mills fields

Euler-Lagrange equation is the (elliptic) Yang-Mills Equation

$$d_A^* F_A = 0 \quad \textbf{(YM)} \qquad \text{and} \qquad d_A F_A = 0 \qquad \textbf{(Bianchi identity)}.$$

- Weak/smooth solutions: **Weak/smooth Yang-Mills fields**;
- Intermediate notion: **Stationary Yang-Mills fields**;
- Special solutions: **ASD or Ω-ASD instantons** (More on bonus slide).

The rainbow in the horizon

Yang-Mills Plateau problem for $n \geq 5$:

Existence of minimizer for $m := \inf \left\{ YM(A) = \int_{M^n} |F_A|^2 : \iota^*_{\partial M^n} A = \eta \right\}$.

Regularity of minimizers and more generally, for stationary connections.

Tian's regularity conjecture for stationary Yang-Mills fields [19]

If A is a stationary YM field with $YM(A) < \infty$, then there exists a closed subset Σ with $\mathcal{H}^{n-5}(\Sigma \cap K) < \infty$ for any $K \subset\subset M^n$ such that in some gauge, A is smooth in $M^n \setminus \Sigma$.

Stationary Yang-Mills fields

A critical point A of YM on $B_1^n(0) \times G$ is **stationary** if for all vector fields $X \in C_0^\infty(B_1^n(0); \mathbb{R}^n)$, the flow ϕ_t of χ satisfies $\frac{d}{dt} \int_{B_1^n(0)} |\phi_t^* F_A|^2 \Big|_{t=0} = 0.$

Weak compactness and gauge fixing

Weak compactness for YM energy

$\{(P^{\nu}, A^{\nu})\}_{\nu \geq 1}$ with $YM(A^{\nu})$ is **uniformly bounded.** We want a limiting bundle with a connection (P^{∞}, A^{∞}) such that $YM(A^{\infty}) \leq \liminf YM(A^{\nu})$.

Weak compactness and gauge fixing

Weak compactness for YM energy

$\{(P^{\nu}, A^{\nu})\}_{\nu\geq 1}$ with $YM(A^{\nu})$ is **uniformly bounded.** We want a limiting bundle with a connection (P^{∞}, A^{∞}) such that $YM(A^{\infty}) \leq \liminf YM(A^{\nu})$.

- *YM* is **not** coercive! (gauge invariance)

Weak compactness and gauge fixing

Weak compactness for YM energy

$\{(P^{\nu}, A^{\nu})\}_{\nu \geq 1}$ with $YM(A^{\nu})$ is **uniformly bounded.** We want a limiting bundle with a connection (P^{∞}, A^{∞}) such that $YM(A^{\infty}) \leq \liminf YM(A^{\nu})$.

- YM is **not** coercive! (gauge invariance) **Gauge fixing**.

Weak compactness and gauge fixing

Weak compactness for YM energy

$\{(P^{\nu}, A^{\nu})\}_{\nu \geq 1}$ with $YM(A^{\nu})$ is **uniformly bounded.** We want a limiting bundle with a connection (P^{∞}, A^{∞}) such that $YM(A^{\infty}) \leq \liminf YM(A^{\nu})$.

- YM is **not** coercive! (gauge invariance) **Gauge fixing**.
  Gaffney inequality – controlling $\|dA\|_{L^p}$ and $\|d^*A\|_{L^p}$ controls $\|\nabla A\|_{L^p}$.

Weak compactness and gauge fixing

Weak compactness for YM energy

$\{(P^\nu, A^\nu)\}_{\nu \geq 1}$ with $YM(A^\nu)$ is **uniformly bounded.** We want a limiting bundle with a connection (P^∞, A^∞) such that $YM(A^\infty) \leq \liminf YM(A^\nu)$.

- *YM* is **not** coercive! (gauge invariance) **Gauge fixing**.
  Gaffney inequality – controlling $\|dA\|_{L^p}$ and $\|d^*A\|_{L^p}$ controls $\|\nabla A\|_{L^p}$.
  Use gauge freedom to choose *local* **Coulomb gauges**, i.e.

$$d^* A_\alpha^{\rho_\alpha} = d^*\left(\rho_\alpha^{-1} d\rho_\alpha + \rho_\alpha^{-1} A_\alpha \rho_\alpha\right) = 0 \qquad \text{in } U_\alpha.$$

Weak compactness and gauge fixing

Weak compactness for YM energy

$\{(P^{\nu}, A^{\nu})\}_{\nu \geq 1}$ with $YM(A^{\nu})$ is **uniformly bounded.** We want a limiting bundle with a connection (P^{∞}, A^{∞}) such that $YM(A^{\infty}) \leq \liminf YM(A^{\nu})$.

- YM is **not** coercive! (gauge invariance) **Gauge fixing**.
  Gaffney inequality – controlling $\|dA\|_{L^p}$ and $\|d^* A\|_{L^p}$ controls $\|\nabla A\|_{L^p}$.
  Use gauge freedom to choose *local* **Coulomb gauges**, i.e.

$$d^* A_{\alpha}^{\rho_\alpha} = d^* \left(\rho_{\alpha}^{-1} d\rho_{\alpha} + \rho_{\alpha}^{-1} A_{\alpha} \rho_{\alpha} \right) = 0 \qquad \text{in } U_{\alpha}.$$

- $F_A \in L^2$ implies, at best, $A \in W^{1,2}$. By Sobolev embedding, $W^{1,2} \hookrightarrow L^{\frac{2n}{n-2}}$.

Weak compactness and gauge fixing

Weak compactness for YM energy

$\{(P^{\nu}, A^{\nu})\}_{\nu \geq 1}$ with $YM(A^{\nu})$ is **uniformly bounded.** We want a limiting bundle with a connection (P^{∞}, A^{∞}) such that $YM(A^{\infty}) \leq \liminf YM(A^{\nu})$.

- YM is **not** coercive! (gauge invariance) **Gauge fixing**.
  Gaffney inequality – controlling $\|dA\|_{L^p}$ and $\|d^*A\|_{L^p}$ controls $\|\nabla A\|_{L^p}$.
  Use gauge freedom to choose *local* **Coulomb gauges**, i.e.

$$d^* A_{\alpha}^{\rho_{\alpha}} = d^* \left(\rho_{\alpha}^{-1} d\rho_{\alpha} + \rho_{\alpha}^{-1} A_{\alpha} \rho_{\alpha} \right) = 0 \qquad \text{in } U_{\alpha}.$$

- $F_A \in L^2$ implies, at best, $A \in W^{1,2}$. By Sobolev embedding, $W^{1,2} \hookrightarrow L^{\frac{2n}{n-2}}$.
  Thus for $n \leq 3$, **the subcritical dimensions**, the quadratic term $A \wedge A$ is a compact perturbation.

Weak compactness and gauge fixing

Weak compactness for YM energy

$\{(P^{\nu}, A^{\nu})\}_{\nu \geq 1}$ with $YM(A^{\nu})$ is **uniformly bounded.** We want a limiting bundle with a connection (P^{∞}, A^{∞}) such that $YM(A^{\infty}) \leq \liminf YM(A^{\nu})$.

- YM is **not** coercive! (gauge invariance) **Gauge fixing**.
  Gaffney inequality – controlling $\|dA\|_{L^p}$ and $\|d^*A\|_{L^p}$ controls $\|\nabla A\|_{L^p}$.
  Use gauge freedom to choose *local* **Coulomb gauges**, i.e.

$$d^* A_{\alpha}^{\rho_{\alpha}} = d^* \left(\rho_{\alpha}^{-1} d\rho_{\alpha} + \rho_{\alpha}^{-1} A_{\alpha} \rho_{\alpha} \right) = 0 \qquad \text{in } U_{\alpha}.$$

- $F_A \in L^2$ implies, at best, $A \in W^{1,2}$. By Sobolev embedding, $W^{1,2} \hookrightarrow L^{\frac{2n}{n-2}}$. Thus for $n \leq 3$, **the subcritical dimensions**, the quadratic term $A \wedge A$ is a compact perturbation. For the **critical dimension** $n = 4$, $W^{1,2} \hookrightarrow L^4$, but the embedding is not compact.

Weak compactness and gauge fixing

Weak compactness for YM energy

$\{(P^{\nu}, A^{\nu})\}_{\nu \geq 1}$ with $YM(A^{\nu})$ is **uniformly bounded.** We want a limiting bundle with a connection (P^{∞}, A^{∞}) such that $YM(A^{\infty}) \leq \liminf YM(A^{\nu})$.

- YM is **not** coercive! (gauge invariance) **Gauge fixing**.
  Gaffney inequality – controlling $\|dA\|_{L^p}$ and $\|d^*A\|_{L^p}$ controls $\|\nabla A\|_{L^p}$.
  Use gauge freedom to choose *local* **Coulomb gauges**, i.e.

$$d^* A_{\alpha}^{\rho_{\alpha}} = d^* \left(\rho_{\alpha}^{-1} d\rho_{\alpha} + \rho_{\alpha}^{-1} A_{\alpha} \rho_{\alpha} \right) = 0 \qquad \text{in } U_{\alpha}.$$

- $F_A \in L^2$ implies, at best, $A \in W^{1,2}$. By Sobolev embedding, $W^{1,2} \hookrightarrow L^{\frac{2n}{n-2}}$. Thus for $n \leq 3$, **the subcritical dimensions**, the quadratic term $A \wedge A$ is a compact perturbation. For the **critical dimension** $n = 4$, $W^{1,2} \hookrightarrow L^4$, but the embedding is not compact. For $n \geq 5$, **the supercritical dimensions**, the L^2 norm of $A \wedge A$ can not be controlled at all by the $W^{1,2}$ norm of A.

Uhlenbeck solved **both problems** for $n = 4$ under the assumption of small energy.

Theorem (Coulomb gauges in the critical dimension, Uhlenbeck '82 [20])

There exists $\varepsilon_{Uh} = \varepsilon_{Uh}(G) > 0$, *such that if* $A \in W^{1,2}\left(B^4_R \times G\right)$ *and* $\|F_A\|_{L^2\left(B^4_R;\Lambda^2\mathbb{R}^4\otimes\mathfrak{g}\right)} < \varepsilon_{Uh}$, *then there exists* $\rho \in W^{2,2}\left(B^4_R; G\right)$ *such that*

$$d^*A^\rho = 0 \quad \text{in } B^4_R, \qquad\qquad \iota^*_{\partial B^4_R}\left(*A^\rho\right) = 0 \quad \text{on } \partial B^4_R.$$

and $C_{Coulomb} \geq 1$ *such that we have the scale-invariant estimate*

$$\|\nabla A^\rho\|_{L^2\left(B^4_R;\mathbb{R}^{n\times n}\otimes\mathfrak{g}\right)} + \|A^\rho\|_{L^4\left(B^4_R;\Lambda^1\mathbb{R}^4\otimes\mathfrak{g}\right)} \leq C_{Coulomb}\,\|F_A\|_{L^2\left(B^4_R;\Lambda^2\mathbb{R}^4\otimes\mathfrak{g}\right)}.$$

Uhlenbeck solved **both problems** for $n = 4$ under the assumption of small energy.

Theorem (Coulomb gauges in the critical dimension, Uhlenbeck '82 [20])

There exists $\varepsilon_{Uh} = \varepsilon_{Uh}(G) > 0$, *such that if* $A \in W^{1,2}\left(B^4_R \times G\right)$ *and* $\|F_A\|_{L^2\left(B^4_R;\Lambda^2\mathbb{R}^4\otimes\mathfrak{g}\right)} < \varepsilon_{Uh}$, *then there exists* $\rho \in W^{2,2}\left(B^4_R; G\right)$ *such that*

$$d^* A^\rho = 0 \quad \text{in } B^4_R, \qquad\qquad \iota^*_{\partial B^4_R}(*A^\rho) = 0 \quad \text{on } \partial B^4_R.$$

and $C_{Coulomb} \geq 1$ *such that we have the scale-invariant estimate*

$$\|\nabla A^\rho\|_{L^2\left(B^4_R;\mathbb{R}^{n\times n}\otimes\mathfrak{g}\right)} + \|A^\rho\|_{L^4\left(B^4_R;\Lambda^1\mathbb{R}^4\otimes\mathfrak{g}\right)} \leq C_{Coulomb}\|F_A\|_{L^2\left(B^4_R;\Lambda^2\mathbb{R}^4\otimes\mathfrak{g}\right)}.$$

The gauge changes are only $W^{2,2}$ and need not be continuous. For sequences of **smooth YM fields** on a smooth bundle P with uniformly bopunded YM energy, we can extract weak limiting connection A^∞, but **bubbling** occurs , i.e. **energy can concentrate** on a finite discrete set (Sedlacek [14]). Using Uhlenbeck's **removable singularity** result, there is a smooth **limiting bundle** P^∞, but P^∞ can be topologically different. (Freed-Uhlenbeck [4], Taubes [18], Lawson [8]).

Does topology need continuity?

Brouwer Degree

$u : S^n \to S^n$ is continuous. How 'many' such 'distinct' maps are there?

Does topology need continuity?

Brouwer Degree

$u : S^n \to S^n$ is continuous. How 'many' such 'distinct' maps are there?

- One for every integer. Every map is assigned an integer, called the degree.
- Computes $\pi_n(S^n) = \mathbb{Z}$.

Does topology need continuity?

Brouwer Degree

$u : S^n \to S^n$ is continuous. How ‘many’ such ‘distinct’ maps are there?

- One for every integer. Every map is assigned an integer, called the degree.
- Computes $\pi_n(S^n) = \mathbb{Z}$.

Can a ‘$W^{1,p}$ map’ have a degree? $u \in W^{1,p} \approx u \in L^p, \nabla u \in L^p.(1 < p < \infty)$

Does topology need continuity?

Brouwer Degree

$u : S^n \to S^n$ is continuous. How 'many' such 'distinct' maps are there?

- One for every integer. Every map is assigned an integer, called the degree.
- Computes $\pi_n (S^n) = \mathbb{Z}$.

Can a '$W^{1,p}$ map' have a degree? $u \in W^{1,p} \approx u \in L^p, \nabla u \in L^p.(1 < p < \infty)$
Yes if $p > n$ (Sobolev-Morrey embedding: $u \in W^{1,p} \approx u \in C^{0,1-\frac{n}{p}}$).

Does topology need continuity?

Brouwer Degree

$u : S^n \to S^n$ is continuous. How 'many' such 'distinct' maps are there?

- One for every integer. Every map is assigned an integer, called the degree.
- Computes $\pi_n(S^n) = \mathbb{Z}$.

Can a '$W^{1,p}$ map' have a degree? $u \in W^{1,p} \approx u \in L^p, \nabla u \in L^p.(1 < p < \infty)$
Yes if $p > n$ (Sobolev-Morrey embedding: $u \in W^{1,p} \approx u \in C^{0,1-\frac{n}{p}}$).

Degree $p = n$ and beyond (Schoen-Uhlenbeck [13], Brezis-Nirenberg [2])

- $W^{1,n}$ maps have a degree. More generally,
- VMO maps have a degree. $u \in \mathrm{VMO} \approx \lim_{r\to 0} \eta_u(r) = 0$, where

$$\eta_u(r) := \sup_{x, 0<\rho<r} \frac{1}{\rho^n} \int_{B(x,\rho)} \left| u - (u)_{B(x,\rho)} \right| .$$

Does topology need continuity?

Brouwer Degree

$u : S^n \to S^n$ is continuous. How 'many' such 'distinct' maps are there?

- One for every integer. Every map is assigned an integer, called the degree.
- Computes $\pi_n(S^n) = \mathbb{Z}$.

Can a '$W^{1,p}$ map' have a degree? $u \in W^{1,p} \approx u \in L^p, \nabla u \in L^p.(1 < p < \infty)$
Yes if $p > n$ (Sobolev-Morrey embedding: $u \in W^{1,p} \approx u \in C^{0,1-\frac{n}{p}}$).

Degree $p = n$ and beyond (Schoen-Uhlenbeck [13], Brezis-Nirenberg [2])

- $W^{1,n}$ maps have a degree. More generally,
- VMO maps have a degree. $u \in \mathrm{VMO} \approx \lim_{r\to 0} \eta_u(r) = 0$, where

$$\eta_u(r) := \sup_{x, 0<\rho<r} \frac{1}{\rho^n} \int_{B(x,\rho)} \left| u - (u)_{B(x,\rho)} \right|.$$

Both results are proved by approximation by smooth maps.

Manifold-valued maps

Sobolev maps between manifolds

M^n, N^m smooth, Riemannian, N^m cpt with $\partial N^m = \emptyset$, $N^m \hookrightarrow \mathbb{R}^l$ iso. (Nash).

$$u \in W^{1,p}(M^n; N^m) := \left\{ u \in W^{1,p}\left(M^n; \mathbb{R}^l\right), u(x) \in N^m \text{ for a.e. } x \in M^n \right\}. \quad (7)$$

Manifold-valued maps

Sobolev maps between manifolds

M^n, N^m smooth, Riemannian, N^m cpt with $\partial N^m = \emptyset$, $N^m \hookrightarrow \mathbb{R}^l$ iso. (Nash).

$$u \in W^{1,p}(M^n; N^m) := \left\{ u \in W^{1,p}\left(M^n; \mathbb{R}^l\right), u(x) \in N^m \text{ for a.e. } x \in M^n \right\}. \quad (7)$$

$u \in W^{1,p}(B_1^n; \mathbb{R}^m)$. Find smooth $u^\nu \overset{W^{1,p}}{\to} u$. Mollify! works for any finite dim. linear space target V.

Manifold-valued maps

Sobolev maps between manifolds

M^n, N^m smooth, Riemannian, N^m cpt with $\partial N^m = \emptyset$, $N^m \hookrightarrow \mathbb{R}^l$ iso. (Nash).

$$u \in W^{1,p}\left(M^n; N^m\right) := \left\{u \in W^{1,p}\left(M^n; \mathbb{R}^l\right), u(x) \in N^m \text{ for a.e. } x \in M^n\right\}. \quad (7)$$

$u \in W^{1,p}\left(B_1^n; \mathbb{R}^m\right)$. Find smooth $u^\nu \overset{W^{1,p}}{\to} u$. Mollify! works for any finite dim. linear space target V. $W^{1,p}\left(B_1^n; N^m\right)$ is a different animal altogether.

Manifold-valued maps

Sobolev maps between manifolds

M^n, N^m smooth, Riemannian, N^m cpt with $\partial N^m = \emptyset$, $N^m \hookrightarrow \mathbb{R}^l$ iso. (Nash).

$$u \in W^{1,p}\left(M^n; N^m\right) := \left\{u \in W^{1,p}\left(M^n; \mathbb{R}^l\right), u(x) \in N^m \text{ for a.e. } x \in M^n\right\}. \quad (7)$$

$u \in W^{1,p}\left(B_1^n; \mathbb{R}^m\right)$. Find smooth $u^\nu \stackrel{W^{1,p}}{\to} u$. Mollify! works for any finite dim. linear space target V. $W^{1,p}\left(B_1^n; N^m\right)$ is a different animal altogether.

Density in Sobolev spaces of manifold-valued maps

(7) is **not** a linear space!

Manifold-valued maps

Sobolev maps between manifolds

M^n, N^m smooth, Riemannian, N^m cpt with $\partial N^m = \emptyset$, $N^m \hookrightarrow \mathbb{R}^l$ iso. (Nash).

$$u \in W^{1,p}(M^n; N^m) := \left\{ u \in W^{1,p}\left(M^n; \mathbb{R}^l\right), u(x) \in N^m \text{ for a.e. } x \in M^n \right\}. \quad (7)$$

$u \in W^{1,p}(B_1^n; \mathbb{R}^m)$. Find smooth $u^\nu \stackrel{W^{1,p}}{\to} u$. Mollify! works for any finite dim. linear space target V. $W^{1,p}(B_1^n; N^m)$ is a different animal altogether.

Density in Sobolev spaces of manifold-valued maps

(7) is **not** a linear space! u^ν need not be N^m-valued.

Manifold-valued maps

Sobolev maps between manifolds

M^n, N^m smooth, Riemannian, N^m cpt with $\partial N^m = \emptyset$, $N^m \hookrightarrow \mathbb{R}^l$ iso. (Nash).

$$u \in W^{1,p}\left(M^n; N^m\right) := \left\{u \in W^{1,p}\left(M^n; \mathbb{R}^l\right), u(x) \in N^m \text{ for a.e. } x \in M^n\right\}. \quad (7)$$

$u \in W^{1,p}\left(B_1^n; \mathbb{R}^m\right)$. Find smooth $u^\nu \overset{W^{1,p}}{\to} u$. Mollify! works for any finite dim. linear space target V. $W^{1,p}\left(B_1^n; N^m\right)$ is a different animal altogether.

Density in Sobolev spaces of manifold-valued maps

(7) is **not** a linear space! u^ν need not be N^m-valued.
Density in $W^{1,p}\left(B_1^n; N\right)$?

Manifold-valued maps

Sobolev maps between manifolds

M^n, N^m smooth, Riemannian, N^m cpt with $\partial N^m = \emptyset$, $N^m \hookrightarrow \mathbb{R}^l$ iso. (Nash).

$$u \in W^{1,p}(M^n; N^m) := \left\{ u \in W^{1,p}\left(M^n; \mathbb{R}^l\right), u(x) \in N^m \text{ for a.e. } x \in M^n \right\}. \quad (7)$$

$u \in W^{1,p}(B_1^n; \mathbb{R}^m)$. Find smooth $u^\nu \stackrel{W^{1,p}}{\rightarrow} u$. Mollify! works for any finite dim. linear space target V. $W^{1,p}(B_1^n; N^m)$ is a different animal altogether.

Density in Sobolev spaces of manifold-valued maps

(7) is **not** a linear space! u^ν need not be N^m-valued.
Density in $W^{1,p}(B_1^n; N)$?

- **Subcritical** $p > n$, Yes. Continuity is important. Sobolev-Morrey embedding

Manifold-valued maps

Sobolev maps between manifolds

M^n, N^m smooth, Riemannian, N^m cpt with $\partial N^m = \emptyset$, $N^m \hookrightarrow \mathbb{R}^l$ iso. (Nash).

$$u \in W^{1,p}\left(M^n; N^m\right) := \left\{u \in W^{1,p}\left(M^n; \mathbb{R}^l\right), u(x) \in N^m \text{ for a.e. } x \in M^n\right\}. \quad (7)$$

$u \in W^{1,p}\left(B_1^n; \mathbb{R}^m\right)$. Find smooth $u^\nu \overset{W^{1,p}}{\to} u$. Mollify! works for any finite dim. linear space target V. $W^{1,p}\left(B_1^n; N^m\right)$ is a different animal altogether.

Density in Sobolev spaces of manifold-valued maps

(7) is **not** a linear space! u^ν need not be N^m-valued.
Density in $W^{1,p}\left(B_1^n; N\right)$?

- **Subcritical** $p > n$, Yes. Continuity is important. Sobolev-Morrey embedding
- **Critical** $p = n$, Still **yes**! Schoen-Uhlenbeck [13]. Harmonic maps.

Manifold-valued maps

Sobolev maps between manifolds

M^n, N^m smooth, Riemannian, N^m cpt with $\partial N^m = \emptyset$, $N^m \hookrightarrow \mathbb{R}^l$ iso. (Nash).

$$u \in W^{1,p}\left(M^n; N^m\right) := \left\{ u \in W^{1,p}\left(M^n; \mathbb{R}^l\right), u(x) \in N^m \text{ for a.e. } x \in M^n \right\}. \quad (7)$$

$u \in W^{1,p}\left(B_1^n; \mathbb{R}^m\right)$. Find smooth $u^\nu \overset{W^{1,p}}{\to} u$. Mollify! works for any finite dim. linear space target V. $W^{1,p}\left(B_1^n; N^m\right)$ is a different animal altogether.

Density in Sobolev spaces of manifold-valued maps

(7) is **not** a linear space! u^ν need not be N^m-valued.
Density in $W^{1,p}\left(B_1^n; N\right)$?

- **Subcritical** $p > n$, Yes. Continuity is important. Sobolev-Morrey embedding
- **Critical** $p = n$, Still **yes**! Schoen-Uhlenbeck [13]. Harmonic maps.
- **Supercritical** $p < n$, **No!**. Yes iff $\pi_{\lfloor p \rfloor}\left(N^m\right) = 0$. Bethuel [1], Hang-Lin [5].

Approximation questions in gauge theory context

Questions

- Is there an analogue of approximation result of $W^{1,n}$ maps in our context?
- Is there an analogue of approximation result of VMO maps in our context?
- Can we define a notion of bundle topology in those settings?
- What could be the analogue of the result in the supercritical case?

Results

- The answer to all except the last one is **Yes**.
- **Sobolev case:** The first one corresponds to approximation of $W^{1,4}$ bundles with $\mathcal{U}^{1,4}$ connections (locally $A \in L^4$, $dA \in L^2$).
- **Vanishing Morrey-Sobolev case:** The second one corresponds to approximation of $W^{1,\mathrm{VL}^{4,n-4}}$ bundles with $\mathrm{V}\mathcal{U}^{1,\mathrm{L}^{4,n-4}}$ connections (locally $A \in \mathrm{VL}^{4,n-4}$, $dA \in \mathrm{VL}^{2,n-4}$).
- The last one is not fully settled yet, but work in progress.

Monotonicity and Morrey norms

Theorem (Monotonicity formula, Price '83, [11])

Let A be a **stationary** *YM connection on the trivial bundle $B_1^n(0) \times G$, then for any x, r such that $B_r^n(x) \subset\subset B_1^n(0)$, we have*

$$\frac{d}{dr}\left(\frac{1}{r^{n-4}}\int_{B_r^n(x)} |F_A|^2\right) \geq 0.$$

Monotonicity $\Rightarrow$ L^2 bounds on the curvature implies $\mathrm{L}^{2,n-4}$ bounds.

Monotonicity and Morrey norms

Theorem (Monotonicity formula, Price '83, [11])

Let A be a **stationary** *YM connection on the trivial bundle $B_1^n(0) \times G$, then for any x, r such that $B_r^n(x) \subset\subset B_1^n(0)$, we have*

$$\frac{d}{dr}\left(\frac{1}{r^{n-4}}\int_{B_r^n(x)} |F_A|^2\right) \geq 0.$$

Monotonicity $\Rightarrow L^2$ bounds on the curvature implies $\mathrm{L}^{2,n-4}$ bounds.

Morrey spaces

$$\text{For } 0 \leq \lambda < n,\ u \in \mathrm{L}^{p,\lambda}(B_1^n) \quad \approx \quad \|u\|^p_{\mathrm{L}^{p,\lambda}} := \sup_{x,r} \frac{1}{r^\lambda}\int_{B_r^n(x)} |u|^p < \infty.$$

Morrey-Sobolev spaces: $u \in W^{1,\mathrm{L}^{p,\lambda}} \approx u \in \mathrm{L}^{p,\lambda}, \nabla u \in \mathrm{L}^{p,\lambda}$.

Sobolev embeddings in Morrey-Sobolev spaces

Theorem (Adams)

$\Omega \subset \mathbb{R}^n$ *bounded, open, smooth,* $1 < p < n$ *and* $0 \leq \lambda < n - p$, *then we have,*

$$u \in W^{1,\mathrm{L}^{p,\lambda}}(\Omega) \Rightarrow u \in \mathrm{L}^{\frac{(n-\lambda)p}{n-\lambda-p},\lambda}(\Omega).$$

Sobolev embeddings in Morrey-Sobolev spaces

Theorem (Adams)

$\Omega \subset \mathbb{R}^n$ bounded, open, smooth, $1 < p < n$ and $0 \leq \lambda < n - p$, then we have,

$$u \in W^{1,\mathrm{L}^{p,\lambda}}(\Omega) \Rightarrow u \in \mathrm{L}^{\frac{(n-\lambda)p}{n-\lambda-p},\lambda}(\Omega).$$

- For $\lambda = n - 4$, the integrability exponent is $4p/(4-p)$, i.e. p^* in dimension 4.

Vanishing Morrey spaces

$u \in \mathrm{VL}^{p,\lambda}(B_1^n) \quad \approx u \in \mathrm{L}^{p,\lambda}(B_1^n)$ and $\lim_{r\to 0} \eta_u(r) = 0$, where

$$\eta_u(r) := \sup_{x, 0<\rho<r} \frac{1}{\rho^\lambda} \int_{B_\rho^n(x)} |u|^p .$$

Sobolev embeddings in Morrey-Sobolev spaces

Theorem (Adams)

$\Omega \subset \mathbb{R}^n$ bounded, open, smooth, $1 < p < n$ and $0 \leq \lambda < n - p$, then we have,

$$u \in W^{1,\mathrm{L}^{p,\lambda}}(\Omega) \Rightarrow u \in \mathrm{L}^{\frac{(n-\lambda)p}{n-\lambda-p},\lambda}(\Omega).$$

- For $\lambda = n - 4$, the integrability exponent is $4p/(4-p)$, i.e. p^* in dimension 4.

Vanishing Morrey spaces

$u \in \mathrm{VL}^{p,\lambda}(B_1^n) \quad \approx u \in \mathrm{L}^{p,\lambda}(B_1^n)$ and $\lim_{r \to 0} \eta_u(r) = 0$, where

$$\eta_u(r) := \sup_{x, 0<\rho<r} \frac{1}{\rho^\lambda} \int_{B_\rho^n(x)} |u|^p.$$

- For $\lambda = 0$, $\mathrm{VL}^{p,\lambda} = \mathrm{L}^{p,\lambda} = L^p$. For $\lambda > 0$, $\mathrm{VL}^{p,\lambda} \subsetneq \mathrm{L}^{p,\lambda}$ and smooth functions are strongly dense in $\mathrm{VL}^{p,\lambda}$, but not in $\mathrm{L}^{p,\lambda}$.
  $u \in W^{1,\mathrm{VL}^{p,\lambda}} \approx u \in \mathrm{L}^{p,\lambda}, \nabla u \in \mathrm{VL}^{p,\lambda}$.

Vanishing Morrey-Sobolev bundles

Vanishing Morrey-Sobolev bundles

$$P = \left(\{U_\alpha\}_{\alpha \in I}, \{g_{\alpha\beta}\}_{\alpha,\beta \in I} \right) \in \mathrm{V}\mathcal{P}_G^{1,\mathrm{L}^{4,n-4}} \text{ if } g_{\alpha\beta} \in W^{1,\mathrm{VL}^{4,n-4}}(U_\alpha \cap U_\beta; G)$$

for all $\alpha, \beta \in I$ with nonempty intersection and satisfies the cocycle condition

$$g_{\alpha\beta}(x) g_{\beta\gamma}(x) = g_{\alpha\gamma}(x) \qquad \text{for a.e. } x \in U_\alpha \cap U_\beta \cap U_\gamma. \tag{8}$$

Vanishing Morrey-Sobolev bundles

Vanishing Morrey-Sobolev bundles

$$P = \left(\{U_\alpha\}_{\alpha \in I}, \{g_{\alpha\beta}\}_{\alpha,\beta \in I} \right) \in \mathrm{V}\mathcal{P}_G^{1,\mathrm{L}^{4,n-4}} \text{ if } g_{\alpha\beta} \in W^{1,\mathrm{VL}^{4,n-4}}(U_\alpha \cap U_\beta; G)$$

for all $\alpha, \beta \in I$ with nonempty intersection and satisfies the cocycle condition

$$g_{\alpha\beta}(x) g_{\beta\gamma}(x) = g_{\alpha\gamma}(x) \qquad \text{for a.e. } x \in U_\alpha \cap U_\beta \cap U_\gamma. \tag{8}$$

Possible Strategy

Approximate $g_{\alpha\beta}$. Poincaré inequality and Hölder inequality imply

$$\frac{1}{\rho^n} \int_{B(x,\rho)} \left| u - (u)_{B(x,\rho)} \right| \leq \frac{1}{\rho^{n-1}} \int_{B(x,\rho)} |\nabla u| \leq \left(\frac{1}{\rho^{n-4}} \int_{B(x,\rho)} |\nabla u|^4 \right)^{\frac{1}{4}}.$$

So $u \in W^{1,\mathrm{VL}^{4,n-4}}(\Omega) \Rightarrow u \in \mathrm{VMO}(\Omega)$.

Vanishing Morrey-Sobolev bundles

Vanishing Morrey-Sobolev bundles

$$P = \left(\{U_\alpha\}_{\alpha \in I}, \{g_{\alpha\beta}\}_{\alpha,\beta \in I} \right) \in \mathrm{V}\mathcal{P}_G^{1,\mathrm{L}^{4,n-4}} \text{ if } g_{\alpha\beta} \in W^{1,\mathrm{VL}^{4,n-4}}\left(U_\alpha \cap U_\beta; G\right)$$

for all $\alpha, \beta \in I$ with nonempty intersection and satisfies the cocycle condition

$$g_{\alpha\beta}(x) g_{\beta\gamma}(x) = g_{\alpha\gamma}(x) \qquad \text{for a.e. } x \in U_\alpha \cap U_\beta \cap U_\gamma. \tag{8}$$

Possible Strategy

Approximate $g_{\alpha\beta}$. Poincaré inequality and Hölder inequality imply

$$\frac{1}{\rho^n} \int_{B(x,\rho)} \left| u - (u)_{B(x,\rho)} \right| \leq \frac{1}{\rho^{n-1}} \int_{B(x,\rho)} |\nabla u| \leq \left(\frac{1}{\rho^{n-4}} \int_{B(x,\rho)} |\nabla u|^4 \right)^{\frac{1}{4}}.$$

So $u \in W^{1,\mathrm{VL}^{4,n-4}}(\Omega) \Rightarrow u \in \mathrm{VMO}(\Omega)$. Difficulty: cocycle conditions (8).

Approximation in the supercritical dimension

Use connections and Coulomb gauges

$A \in \mathrm{V}\mathcal{U}^{1,\mathrm{L}^{4,n-4}}$ means $A_\alpha \in \mathrm{VL}^{4,n-4}$ and $dA_\alpha \in \mathrm{VL}^{2,n-4}$ for every α. This is the minimum assumption for $F_{A_\alpha} \in \mathrm{VL}^{2,n-4}$.

Approximation in the supercritical dimension

Use connections and Coulomb gauges

$A \in \mathrm{V}\mathcal{U}^{1,\mathrm{L}^{4,n-4}}$ means $A_\alpha \in \mathrm{VL}^{4,n-4}$ and $dA_\alpha \in \mathrm{VL}^{2,n-4}$ for every α. This is the minimum assumption for $F_{A_\alpha} \in \mathrm{VL}^{2,n-4}$.

Only elliptic estimates, easy in Abelian case, new point of view towards topology.

Approximation in the supercritical dimension

Use connections and Coulomb gauges

$A \in \mathrm{V}\mathcal{U}^{1,\mathrm{L}^{4,n-4}}$ means $A_{\alpha} \in \mathrm{VL}^{4,n-4}$ and $dA_{\alpha} \in \mathrm{VL}^{2,n-4}$ for every α. This is the minimum assumption for $F_{A_{\alpha}} \in \mathrm{VL}^{2,n-4}$.

Only elliptic estimates, easy in Abelian case, new point of view towards topology.

Theorem (Vanishing Morrey bundles with connections, S. '20, S. '19 [16])

Given any $P \in \mathrm{V}\mathcal{P}_{G}^{1,\mathrm{L}^{4,n-4}}(M^n)$ and $A \in \mathrm{V}\mathcal{U}^{1,\mathrm{L}^{4,n-4}}(P)$, there is a sequence of smooth principal G-bundles $P^{\nu} \in \mathcal{P}_{G}^{\infty}(M^n)$ with smooth connections $A^{\nu} \in \mathcal{A}^{\infty}(P^{\nu})$ such that $P^{\nu} \overset{W^{1,\mathrm{L}^{4,n-4}}}{\simeq_{\rho^{\nu}}} P$ and for all i,

$$g_{ij}^{\nu} \to g_{ij} \quad \text{in } W^{1,\mathrm{L}^{4,n-4}} \qquad \text{and} \qquad A_i^{\nu} - (\rho_i^{\nu})^{*} A_i \to 0 \quad \text{in } \mathcal{U}^{1,\mathrm{L}^{4,n-4}}.$$

Approximation in the supercritical dimension

Use connections and Coulomb gauges

$A \in \mathrm{V}\mathcal{U}^{1,\mathrm{L}^{4,n-4}}$ means $A_{\alpha} \in \mathrm{VL}^{4,n-4}$ and $dA_{\alpha} \in \mathrm{VL}^{2,n-4}$ for every α. This is the minimum assumption for $F_{A_{\alpha}} \in \mathrm{VL}^{2,n-4}$.

Only elliptic estimates, easy in Abelian case, new point of view towards topology.

Theorem (Vanishing Morrey bundles with connections, S. '20, S. '19 [16])

Given any $P \in \mathrm{V}\mathcal{P}_{G}^{1,\mathrm{L}^{4,n-4}}(M^n)$ and $A \in \mathrm{V}\mathcal{U}^{1,\mathrm{L}^{4,n-4}}(P)$, there is a sequence of smooth principal G-bundles $P^{\nu} \in \mathcal{P}_{G}^{\infty}(M^n)$ with smooth connections $A^{\nu} \in \mathcal{A}^{\infty}(P^{\nu})$ such that $P^{\nu} \overset{W^{1,\mathrm{L}^{4,n-4}}}{\simeq_{\rho^{\nu}}} P$ and for all i,

$$g_{ij}^{\nu} \to g_{ij} \quad \text{in } W^{1,\mathrm{L}^{4,n-4}} \qquad \text{and} \qquad A_i^{\nu} - (\rho_i^{\nu})^{*} A_i \to 0 \quad \text{in } \mathcal{U}^{1,\mathrm{L}^{4,n-4}}.$$

For $n = 4$, this reduces to approximation of $\mathcal{U}^{1,4}$ connections on $W^{1,4}$ bundles. That case is also obtained earlier in Isobe '09 [6] and the Abelian case in Isobe '14 [7], both using approximation results for G-valued Sobolev maps.

Approximation in the supercritical dimension

Theorem (Approximation in the Abelian case, S. '19 [16], Isobe '14 [7])

Let $2 < p < \infty$. Given $P \in \mathcal{P}^{1,p}_{\mathbb{S}^1}(M^n)$ and $A \in \mathcal{U}^{1,p}(P)$ (i.e. $A \in L^p, dA \in L^{\frac{p}{2}}$), there exists $\{(P^\nu, A^\nu)\}_{\nu \geq 1}$, with $P^\nu \in \mathcal{P}^{\infty}_{\mathbb{S}^1}(M^n)$ and $A^\nu \in \mathcal{A}^\infty(P^\nu)$ such that $P^\nu \overset{W^{1,p}}{\simeq}_{\rho^\nu} P$ and $g^\nu_{ij} \to g_{ij}$ in $W^{1,p}$ and $A^\nu_i - (\rho^\nu_i)^* A_i \to 0$ in $\mathcal{U}^{1,p}$.

Approximation in the supercritical dimension

Theorem (Approximation in the Abelian case, S. '19 [16], Isobe '14 [7])

Let $2 < p < \infty$. Given $P \in \mathcal{P}^{1,p}_{\mathbb{S}^1}(M^n)$ and $A \in \mathcal{U}^{1,p}(P)$ (i.e. $A \in L^p, dA \in L^{\frac{p}{2}}$), there exists $\{(P^\nu, A^\nu)\}_{\nu \geq 1}$, with $P^\nu \in \mathcal{P}^{\infty}_{\mathbb{S}^1}(M^n)$ and $A^\nu \in \mathcal{A}^\infty(P^\nu)$ such that $P^\nu \overset{W^{1,p}}{\simeq}_{\rho^\nu} P$ and $g^\nu_{ij} \to g_{ij}$ in $W^{1,p}$ and $A^\nu_i - (\rho^\nu_i)^* A_i \to 0$ in $\mathcal{U}^{1,p}$.

In all cases, approximation of bundles follows. Pick a partition of unity $\{\psi_\alpha\}_\alpha$ subordinate to the cover $\{U_\alpha\}_{\alpha \in I}$ and define

$$A_\alpha := \sum_{\substack{\beta \in I, \beta \neq \alpha, \\ U_\alpha \cap U_\beta \neq \emptyset}} \psi_\beta g^{-1}_{\beta\alpha} dg_{\beta\alpha} \qquad \text{for each } \alpha \in I.$$

But does not exclude the possibility to have **C^0-distinct** sequences of smooth bundles $\{P^\nu_1\}_{\nu \geq 1}, \{P^\nu_2\}_{\nu \geq 1}$, both approximating P.
$(P^\nu_1, A^\nu_1) \to (P, A_1)$ and $(P^\nu_2, A^\nu_2) \to (P, A_2)$.

Approximation result

Main steps of the proof

1. Given (P, A), pass to local Coulomb gauges for A and construct the Coulomb bundles and connection $(P_{A_{Coulomb}}, A_{Coulomb})$ which is $W^{1,\mathrm{VL}^{4,n-4}}$ gauge equivalent to (P, A).
2. Show that $P_{A_{Coulomb}}$ is actually a $W^{2,q} \cap C^{0,\alpha}$-bundle for any $\frac{n}{2} < q < n$ and $\alpha < 1$. (This is the key point).
3. Approximate $P_{A_{Coulomb}}$ by smooth bundles P^{ν}, which are $W^{1,\mathrm{VL}^{4,n-4}}$ (in fact C^0) gauge equivalent (Much cleaner due to the improved regularity).
4. Pull back A on P^{ν}. The pullback is now a vanishing Morrey-Sobolev connection on a *smooth* bundle.
5. Approximate locally by smooth connection forms and glue (a trick).

Approximation result

Main steps of the proof

1. Given (P, A), pass to local Coulomb gauges for A and construct the Coulomb bundles and connection $(P_{A_{Coulomb}}, A_{Coulomb})$ which is $W^{1,\mathrm{VL}^{4,n-4}}$ gauge equivalent to (P, A).
2. Show that $P_{A_{Coulomb}}$ is <u>actually a $W^{2,q} \cap C^{0,\alpha}$-bundle</u> for any $\frac{n}{2} < q < n$ and $\alpha < 1$. (This is the key point).
3. Approximate $P_{A_{Coulomb}}$ by smooth bundles P^{ν}, which are $W^{1,\mathrm{VL}^{4,n-4}}$ (in fact C^0) gauge equivalent (Much cleaner due to the improved regularity).
4. Pull back A on P^{ν}. The pullback is now a vanishing Morrey-Sobolev connection on a *smooth* bundle.
5. Approximate locally by smooth connection forms and glue (a trick).

Remark on Isobe's proof

Since $P \in W^{1,n}$, approximating P directly is messier than Step 3 above. In the Abelian case, needs Bethuel's result and the fact that $\pi_i \left(\mathbb{S}^1\right) = 0$ for all $i \geq 2$.

Local Coulomb gauges in Morrey-Sobolev setting

Theorem (Coulomb gauges, Meyer-Rivière '03 [9], Tao-Tian '04 [17])

There exists $\varepsilon_{Uh} = \varepsilon_{Uh}(n, G) > 0$, *such that if* $A \in \mathrm{V}\mathcal{U}^{1,\mathrm{L}^{4,n-4}}(B^n_R \times G)$ *and* $\|F_A\|_{\mathrm{L}^{2,n-4}(B^n_R;\Lambda^2\mathbb{R}^n\otimes\mathfrak{g})} < \varepsilon_{Uh}$, *then there exists* $\rho \in W^{1,\mathrm{VL}^{4,n-4}}(B^n_R; G)$ *such that*

$$\begin{cases} d^* A^\rho = 0 & \text{in } B^n_R, \\ \iota^*_{\partial B^n_R}(*A^\rho) = 0 & \text{on } \partial B^n_R \end{cases}$$

and $C_{Coulomb} \geq 1$ *and we have the scale-invariant estimate*

$$\|\nabla A^\rho\|_{\mathrm{L}^{2,n-4}} + \|A^\rho\|_{\mathrm{L}^{4,n-4}} \leq C_{Coulomb}\|F_A\|_{\mathrm{L}^{2,n-4}}.$$

Meyer-Rivière [9] and Tao-Tian [17] proved this result for **smooth** connections, which holds for $\mathrm{V}\mathcal{U}^{1,\mathrm{L}^{4,n-4}}$ connections by density in vanishing Morrey-Sobolev spaces.

Regularity of Coulomb bundles

Theorem (Improved regularity of Coulomb bundles, S.'20)

Let $P \in \mathrm{V}\mathcal{P}_G^{1,\mathrm{L}^{4,n-4}}(M^n)$ and $A \in \mathrm{V}\mathcal{U}^{1,\mathrm{L}^{4,n-4}}(P)$ be a connection on P which is Coulomb, then P is a $W^{2,q} \cap C^{0,\alpha}$-bundle for any $\frac{n}{2} < q < n$ and $\alpha < 1$.

Regularity of Coulomb bundles

Theorem (Improved regularity of Coulomb bundles, S.'20)

Let $P \in \mathrm{V}\mathcal{P}_{G}^{1,\mathrm{L}^{4,n-4}}(M^n)$ and $A \in \mathrm{V}\mathcal{U}^{1,\mathrm{L}^{4,n-4}}(P)$ be a connection on P which is Coulomb, then P is a $W^{2,q} \cap C^{0,\alpha}$-bundle for any $\frac{n}{2} < q < n$ and $\alpha < 1$.

- This is in a sense quite striking. Both the bundle we started with and the Coulomb gauges have only Morrey-Sobolev regularity and need not even be continuous. But the Coulomb bundle is much more regular just by virtue of the fact that the connection forms satisfies the Coulomb conditions.
- For $n = 4$, this reduces to the critical dimension case.
- Previous results for $n = 4$: Rivière '02 [12], Shevchishin '02 [15].

Ingredients for the result

Lemma (Elliptic estimate in Morrey-Sobolev setting, S. '20)

$n \geq 3, N \geq 1$ integers, $1 < m \leq \frac{n}{2}$ and $m < p < 2m$, $\Omega \subset \mathbb{R}^n$ bounded, open. Let $B \in \mathrm{L}^{2m,n-2m}\left(\Omega; \Lambda^1\mathbb{R}^n \otimes \mathfrak{M}(N)\right)$ and $f \in \mathrm{L}^{p,n-2m}\left(\Omega; \mathbb{R}^N\right)$. There exists $\varepsilon_{\Delta_{Cr}}$ such that if $\|B\|_{\mathrm{L}^{2m,n-2m}(\Omega;\Lambda^1\mathbb{R}^n\otimes\mathfrak{M}(N))} \leq \varepsilon_{\Delta_{Cr}}$ and $u \in W^{1,\mathrm{L}^{2,n-2m}}\left(\Omega; \mathbb{R}^N\right)$ solves

$$\Delta u = B \cdot \nabla u + f \qquad \text{in } \Omega, \tag{9}$$

then $u \in W^{1,\mathrm{L}^{\frac{2mp}{2m-p},n-2m}}_{loc}\left(\Omega; \mathbb{R}^N\right)$ with corresponding estimates.

Ingredients for the result

Lemma (Elliptic estimate in Morrey-Sobolev setting, S. '20)

$n \geq 3, N \geq 1$ integers, $1 < m \leq \frac{n}{2}$ and $m < p < 2m$, $\Omega \subset \mathbb{R}^n$ bounded, open. Let $B \in \mathrm{L}^{2m,n-2m}\left(\Omega; \Lambda^1\mathbb{R}^n \otimes \mathfrak{M}(N)\right)$ and $f \in \mathrm{L}^{p,n-2m}\left(\Omega; \mathbb{R}^N\right)$. There exists $\varepsilon_{\Delta_{Cr}}$ such that if $\|B\|_{\mathrm{L}^{2m,n-2m}(\Omega;\Lambda^1\mathbb{R}^n\otimes\mathfrak{M}(N))} \leq \varepsilon_{\Delta_{Cr}}$ and $u \in W^{1,\mathrm{L}^{2,n-2m}}\left(\Omega; \mathbb{R}^N\right)$ solves

$$\Delta u = B \cdot \nabla u + f \qquad \text{in } \Omega, \tag{9}$$

then $u \in W^{1,\mathrm{L}^{\frac{2mp}{2m-p},n-2m}}_{loc}\left(\Omega; \mathbb{R}^N\right)$ with corresponding estimates.

From the gluing relation $A_\beta = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta}$, we get,

$$d^* dg_{\alpha\beta} = d^*\left(g_{\alpha\beta}A_\beta - A_\alpha g_{\alpha\beta}\right) \qquad \text{in } U_\alpha \cap U_\beta.$$

Ingredients for the result

Lemma (Elliptic estimate in Morrey-Sobolev setting, S. '20)

$n \geq 3, N \geq 1$ integers, $1 < m \leq \frac{n}{2}$ and $m < p < 2m$, $\Omega \subset \mathbb{R}^n$ bounded, open. Let $B \in \mathrm{L}^{2m,n-2m}\left(\Omega; \Lambda^1\mathbb{R}^n \otimes \mathfrak{M}\left(N\right)\right)$ and $f \in \mathrm{L}^{p,n-2m}\left(\Omega; \mathbb{R}^N\right)$. There exists $\varepsilon_{\Delta_{Cr}}$ such that if $\|B\|_{\mathrm{L}^{2m,n-2m}(\Omega;\Lambda^1\mathbb{R}^n\otimes\mathfrak{M}(N))} \leq \varepsilon_{\Delta_{Cr}}$ and $u \in W^{1,\mathrm{L}^{2,n-2m}}\left(\Omega; \mathbb{R}^N\right)$ solves

$$\Delta u = B \cdot \nabla u + f \qquad \text{in } \Omega, \tag{9}$$

then $u \in W^{1,\mathrm{L}^{\frac{2mp}{2m-p},n-2m}}_{loc}\left(\Omega; \mathbb{R}^N\right)$ with corresponding estimates.

From the gluing relation $A_\beta = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta}$, we get,

$$d^* dg_{\alpha\beta} = d^*\left(g_{\alpha\beta}A_\beta - A_\alpha g_{\alpha\beta}\right) \qquad \text{in } U_\alpha \cap U_\beta.$$

Since A is Coulomb, i.e. $d^*A_\alpha = 0 = d^*A_\beta$ in $U_\alpha \cap U_\beta$, we get,

$$-\Delta g_{\alpha\beta} = *\left[dg_{\alpha\beta} \wedge (*A_\beta)\right] + *\left[(*A_\alpha) \wedge dg_{\alpha\beta}\right] \qquad \text{in } U_\alpha \cap U_\beta.$$

Ingredients for the result

Lemma (Elliptic estimate in Morrey-Sobolev setting, S. '20)

$n \geq 3, N \geq 1$ integers, $1 < m \leq \frac{n}{2}$ and $m < p < 2m$, $\Omega \subset \mathbb{R}^n$ bounded, open. Let $B \in \mathrm{L}^{2m,n-2m}\left(\Omega; \Lambda^1\mathbb{R}^n \otimes \mathfrak{M}\left(N\right)\right)$ and $f \in \mathrm{L}^{p,n-2m}\left(\Omega; \mathbb{R}^N\right)$. There exists $\varepsilon_{\Delta_{Cr}}$ such that if $\|B\|_{\mathrm{L}^{2m,n-2m}(\Omega;\Lambda^1\mathbb{R}^n\otimes\mathfrak{M}(N))} \leq \varepsilon_{\Delta_{Cr}}$ and $u \in W^{1,\mathrm{L}^{2,n-2m}}\left(\Omega; \mathbb{R}^N\right)$ solves

$$\Delta u = B \cdot \nabla u + f \qquad \text{in } \Omega, \tag{9}$$

then $u \in W^{1,\mathrm{L}^{\frac{2mp}{2m-p},n-2m}}_{loc}\left(\Omega; \mathbb{R}^N\right)$ with corresponding estimates.

From the gluing relation $A_\beta = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta}$, we get,

$$d^* dg_{\alpha\beta} = d^*\left(g_{\alpha\beta}A_\beta - A_\alpha g_{\alpha\beta}\right) \quad \text{in } U_\alpha \cap U_\beta.$$

Since A is Coulomb, i.e. $d^*A_\alpha = 0 = d^*A_\beta$ in $U_\alpha \cap U_\beta$, we get,

$$-\Delta g_{\alpha\beta} = *\left[dg_{\alpha\beta} \wedge (*A_\beta)\right] + *\left[(*A_\alpha) \wedge dg_{\alpha\beta}\right] \qquad \text{in } U_\alpha \cap U_\beta.$$

Shrink the sets to get $\mathrm{L}^{2m,n-2m}$ norms small enough.

Topology of weak bundles with connections

Definition of topology, S. '20, also S '19 [16]

Given $P \in \mathrm{V}\mathcal{P}_G^{1,\mathrm{L}^{4,n-4}}(M^n)$ and $A \in \mathrm{V}\mathcal{U}^{1,\mathrm{L}^{4,n-4}}(P)$, one can associate **the** C^0 equivalence class of 'the' associated Coulomb bundle to the **pair** (P, A), which is stable under $W^{1,\mathrm{VL}^{4,n-4}}$ gauge transformations.

Topology of weak bundles with connections

Definition of topology, S. '20, also S '19 [16]

Given $P \in \mathrm{V}\mathcal{P}_{G}^{1,\mathrm{L}^{4,n-4}}(M^n)$ and $A \in \mathrm{V}\mathcal{U}^{1,\mathrm{L}^{4,n-4}}(P)$, one can associate **the** C^0 equivalence class of 'the' associated Coulomb bundle to the **pair** (P, A), which is stable under $W^{1,\mathrm{VL}^{4,n-4}}$ gauge transformations.

Remarks on the definition

- Our 'topology' is encoded in the pair (P, A), **not** to P alone! Stability is also **only** under gauge transformation of **both**!

Topology of weak bundles with connections

Definition of topology, S. '20, also S '19 [16]

Given $P \in \mathrm{V}\mathcal{P}_G^{1,\mathrm{L}^{4,n-4}}(M^n)$ and $A \in \mathrm{V}\mathcal{U}^{1,\mathrm{L}^{4,n-4}}(P)$, one can associate **the** C^0 equivalence class of 'the' associated Coulomb bundle to the **pair** (P, A), which is stable under $W^{1,\mathrm{VL}^{4,n-4}}$ gauge transformations.

Remarks on the definition

- Our 'topology' is encoded in the pair (P, A), **not** to P alone! Stability is also **only** under gauge transformation of **both**! **Puzzling?!!**

Topology of weak bundles with connections

Definition of topology, S. '20, also S '19 [16]

Given $P \in \mathrm{V}\mathcal{P}_G^{1,\mathrm{L}^{4,n-4}}(M^n)$ and $A \in \mathrm{V}\mathcal{U}^{1,\mathrm{L}^{4,n-4}}(P)$, one can associate **the** C^0 equivalence class of 'the' associated Coulomb bundle to the **pair** (P, A), which is stable under $W^{1,\mathrm{VL}^{4,n-4}}$ gauge transformations.

Remarks on the definition

- Our 'topology' is encoded in the pair (P, A), **not** to P alone! Stability is also **only** under gauge transformation of **both**! **Puzzling?!!** For more regular P and A, this assignment is **independent** of A. (also the same as the usual one).

Topology of weak bundles with connections

Definition of topology, S. '20, also S '19 [16]

Given $P \in \mathrm{V}\mathcal{P}_{G}^{1,\mathrm{L}^{4,n-4}}(M^n)$ and $A \in \mathrm{V}\mathcal{U}^{1,\mathrm{L}^{4,n-4}}(P)$, one can associate **the** C^0 equivalence class of 'the' associated Coulomb bundle to the **pair** (P, A), which is stable under $W^{1,\mathrm{VL}^{4,n-4}}$ gauge transformations.

Remarks on the definition

- Our 'topology' is encoded in the pair (P, A), **not** to P alone! Stability is also **only** under gauge transformation of **both**! **Puzzling?!!** For more regular P and A, this assignment is **independent** of A. (also the same as the usual one).
- Earlier attempts by Isobe '09 [6] and Shevchishin '02 [15] in the critical dimension are very different and associates an C^0 class to P alone.

Topology of weak bundles with connections

Definition of topology, S. '20, also S '19 [16]

Given $P \in \mathrm{V}\mathcal{P}_{G}^{1,\mathrm{L}^{4,n-4}}(M^n)$ and $A \in \mathrm{V}\mathcal{U}^{1,\mathrm{L}^{4,n-4}}(P)$, one can associate **the** C^0 equivalence class of 'the' associated Coulomb bundle to the **pair** (P, A), which is stable under $W^{1,\mathrm{VL}^{4,n-4}}$ gauge transformations.

Remarks on the definition

- Our 'topology' is encoded in the pair (P, A), **not** to P alone! Stability is also **only** under gauge transformation of **both**! **Puzzling?!!** For more regular P and A, this assignment is **independent** of A. (also the same as the usual one).
- Earlier attempts by Isobe '09 [6] and Shevchishin '02 [15] in the critical dimension are very different and associates an C^0 class to P alone.
- Since we need to change gauges to obtain estimate for the connections, in a sense the connection can 'drag' the bundle along with it. Our class can keep a track of what the connections are doing.

Naturality of the topological class

As an illustration, we improve Theorem IV.2.,Rivière '02 [12]).

Theorem (Stability of topology w/o concentration, S. '19 [16])

$P^{\nu} \in \mathcal{P}^{1,4}_{G}\left(M^4\right)$, $A^{\nu} \in \mathcal{U}^{1,4}\left(P^{\nu}\right)$ for $\nu \geq 1$ be sequences of bundles with connections trivialized over a common cover such that $YM\left(A^{\nu}\right)$ is ***uniformly bounded*** *and $\left\{\left|F_{A^{\nu}}\right|^2\right\}_{\nu\geq 1}$ is* ***equiintegrable*** *in M^n. Then there exists $P^{\infty} \in \mathcal{P}^{1,4} \cap \mathcal{P}^{0}\left(M^4\right)$, $A^{\infty} \in W^{1,2}\left(P^{\infty}\right)$ and a subsequence $\left\{A^{\nu_s}\right\}_{s\geq 1}$ such that for large enough s, we have $\left[P^{\nu_s}_{A^{\nu_s}_{Coulomb}}\right]_{C^0} = \left[P^{\infty}\right]_{C^0}$ and for every $i \in I$,*

$$\left(A^{\nu_s}_{Coulomb}\right)_i \rightharpoonup A^{\infty}_i \qquad \text{weakly in } W^{1,2}\left(U^{\infty}_i; \Lambda^1 T^* U^{\infty}_i \otimes \mathfrak{g}\right),$$

$$F_{A^{\nu_s}_i} \rightharpoonup F_{A^{\infty}_i} \qquad \text{weakly in } L^2\left(U^{\infty}_i; \Lambda^2 T^* U^{\infty}_i \otimes \mathfrak{g}\right).$$

If $P^{\nu} = P \in \mathcal{P}^{\infty}_{G}\left(M^4\right)$ and $A^{\nu} \in \mathcal{A}^{\infty}(P)$ for all ν, then $[P]_{C^0} = \left[P^{\infty}\right]_{C^0}$.

Naturality of the topological class

As an illustration, we improve Theorem IV.2.,Rivière '02 [12]).

Theorem (Stability of topology w/o concentration, S. '19 [16])

$P^{\nu} \in \mathcal{P}^{1,4}_{G}\left(M^{4}\right)$, $A^{\nu} \in \mathcal{U}^{1,4}\left(P^{\nu}\right)$ for $\nu \geq 1$ be sequences of bundles with connections trivialized over a common cover such that $YM\left(A^{\nu}\right)$ is **uniformly bounded** *and $\left\{ \left| F_{A^{\nu}} \right|^{2} \right\}_{\nu \geq 1}$ is* **equiintegrable** *in M^{n}. Then there exists $P^{\infty} \in \mathcal{P}^{1,4} \cap \mathcal{P}^{0}\left(M^{4}\right)$, $A^{\infty} \in W^{1,2}\left(P^{\infty}\right)$ and a subsequence $\left\{ A^{\nu_{s}} \right\}_{s \geq 1}$ such that for large enough s, we have $\left[P^{\nu_{s}}_{A^{\nu_{s}}_{Coulomb}} \right]_{C^{0}} = \left[P^{\infty} \right]_{C^{0}}$ and for every $i \in I$,*

$$\begin{aligned} \left(A^{\nu_{s}}_{Coulomb}\right)_{i} \rightharpoonup A^{\infty}_{i} \qquad & \text{weakly in } W^{1,2}\left(U^{\infty}_{i}; \Lambda^{1} T^{*} U^{\infty}_{i} \otimes \mathfrak{g}\right), \\ F_{A^{\nu_{s}}_{i}} \rightharpoonup F_{A^{\infty}_{i}} \qquad & \text{weakly in } L^{2}\left(U^{\infty}_{i}; \Lambda^{2} T^{*} U^{\infty}_{i} \otimes \mathfrak{g}\right). \end{aligned}$$

If $P^{\nu} = P \in \mathcal{P}^{\infty}_{G}\left(M^{4}\right)$ and $A^{\nu} \in \mathcal{A}^{\infty}(P)$ for all ν, then $\left[P\right]_{C^{0}} = \left[P^{\infty}\right]_{C^{0}}$.
Rivière [12] needed A^{ν} **to be strongly convergent in** $W^{1,2}$ **and** $d^{*}A^{\nu}$ **to be strongly convergent in the Lorentz space** $L^{(2,1)}$, not gauge-invariant conditions.

Naturality of the topological class

Theorem (Flatness criterion, S '19 [16])

For any cover $\mathcal{U}$ of M^4, there exists a constant $\delta > 0$, depending only on $\mathcal{U}$, M^4 and G such that if P is a $W^{1,4}$ bundle trivialized over $\mathcal{U}$ and A is a $\mathcal{U}^{1,4}$ connection on P, then

$$\text{either} \quad YM(A) > \delta \qquad \text{or} \qquad [P_{A_{Coulomb}}]_{C^0} = \left[P^0\right]_{C^0},$$

where P^0 is a flat C^0 bundle. If M^4 is simply connected, $P^0 = M^4 \times G$.

Proof.

If not, then there exist sequences $P^\nu \in \mathcal{P}_G^{1,4}\left(M^4\right)$, $A^\nu \in \mathcal{U}^{1,4}\left(P^\nu\right)$ for $\nu \geq 1$ trivialized over $\mathcal{U}$ such that $P^\nu_{A^\nu_{Coulomb}}$ is not C^0 equivalent to any flat bundle for any $\nu \geq 1$ and $YM(A^\nu) \to 0$. But then P^∞ is flat and this contradicts the stability. □

This is the usual YM energy gap for smooth connections on smooth bundles.

Approximation by 'Almost' smooth classes

The vanishing Morrey-Sobolev result is clearly the best we can hope for if we insist on approximation by **smooth bundles and connections**.

Approximation by 'Almost' smooth classes

The vanishing Morrey-Sobolev result is clearly the best we can hope for if we insist on approximation by **smooth bundles and connections**.
Idea from Bethuel's proof of strong density in $W^{1,p}$ iff $\pi_{\lfloor p \rfloor}(N^m) = 0$.

Approximation by 'Almost' smooth classes

The vanishing Morrey-Sobolev result is clearly the best we can hope for if we insist on approximation by **smooth bundles and connections**.
Idea from Bethuel's proof of strong density in $W^{1,p}$ iff $\pi_{\lfloor p \rfloor}(N^m) = 0$.

Bethuel's 'almost' smooth maps

$$\mathcal{R}^{p,\infty}(M^n; N^m) := \begin{Bmatrix} u \in W^{1,p}(M^n; N^m) : u \in C^{\infty}_{loc}(M^n \setminus \Sigma; N^m), \Sigma \text{ is} \\ \text{a finite union of } (n - \lfloor p \rfloor - 1)\text{-dimesnional submanifolds.} \end{Bmatrix}$$

Approximation by 'Almost' smooth classes

The vanishing Morrey-Sobolev result is clearly the best we can hope for if we insist on approximation by **smooth bundles and connections**.
Idea from Bethuel's proof of strong density in $W^{1,p}$ iff $\pi_{\lfloor p \rfloor}(N^m) = 0$.

Bethuel's 'almost' smooth maps

$$\mathcal{R}^{p,\infty}(M^n; N^m) := \left\{ \begin{matrix} u \in W^{1,p}(M^n; N^m) : u \in C^{\infty}_{loc}(M^n \setminus \Sigma; N^m), \Sigma \text{ is} \\ \text{a finite union of } (n - \lfloor p \rfloor - 1)\text{-dimesnional submanifolds.} \end{matrix} \right\}$$

Theorem (Bethuel [1], Hang-Lin [5])

$\mathcal{R}^{p,\infty}(M^n; N^m)$ is dense in $W^{1,p}(M^n; N^m)$.

Approximation by 'Almost' smooth classes

The vanishing Morrey-Sobolev result is clearly the best we can hope for if we insist on approximation by **smooth bundles and connections**.
Idea from Bethuel's proof of strong density in $W^{1,p}$ iff $\pi_{\lfloor p \rfloor}(N^m) = 0$.

Bethuel's 'almost' smooth maps

$$\mathcal{R}^{p,\infty}(M^n; N^m) := \left\{ \begin{array}{l} u \in W^{1,p}(M^n; N^m) : u \in C^{\infty}_{loc}(M^n \setminus \Sigma; N^m), \Sigma \text{ is} \\ \text{a finite union of } (n - \lfloor p \rfloor - 1)\text{-dimesnional submanifolds.} \end{array} \right\}$$

Theorem (Bethuel [1], Hang-Lin [5])

$\mathcal{R}^{p,\infty}(M^n; N^m)$ *is dense in* $W^{1,p}(M^n; N^m)$.

In terms of local pictures around the singular set, restriction of such maps to the sphere $\mathbb{S}^{\lfloor p \rfloor} \subset \mathbb{R}^{\lfloor p \rfloor + 1}$ in the plane transversal to Σ, can realize nontrivial homotopy classes $v : \mathbb{S}^{\lfloor p \rfloor} \to N^m$.

Conjecture about density of almost smooth bundles

'Almost' smooth bundles of Petrache-Rivière [10]

$$\mathcal{R}^{\infty}\left(M^{n}, \mathrm{SU}(2)\right) := \left\{ \begin{array}{c} (P, A) : A \in \mathcal{A}^{\infty}\left(P\right), P \in \mathcal{P}^{\infty}_{\mathrm{SU}(2)}\left(M^{n} \setminus \Sigma\right) \text{ where } \Sigma \text{ is} \\ \text{a finite union of } (n-5)\text{-dimensional submanifolds.} \end{array} \right\}$$

Conjecture about density of almost smooth bundles

'Almost' smooth bundles of Petrache-Rivière [10]

$$\mathcal{R}^{\infty}\left(M^{n}, \mathrm{SU}(2)\right) := \left\{ \begin{array}{c} (P, A) : A \in \mathcal{A}^{\infty}(P), P \in \mathcal{P}^{\infty}_{\mathrm{SU}(2)}\left(M^{n} \setminus \Sigma\right) \text{ where } \Sigma \text{ is} \\ \text{a finite union of } (n-5)\text{-dimensional submanifolds.} \end{array} \right\}$$

Locally, restriction of the bundle to the sphere $\mathbb{S}^4 \subset \mathbb{R}^5$ in the 5-plane transversal to Σ, can have nontrivial second Chern class c_2.

Conjecture about density of almost smooth bundles

'Almost' smooth bundles of Petrache-Rivière [10]

$$\mathcal{R}^{\infty}\left(M^{n},\mathrm{SU}(2)\right) := \left\{ \begin{array}{c} (P,A) : A \in \mathcal{A}^{\infty}\left(P\right), P \in \mathcal{P}^{\infty}_{\mathrm{SU}(2)}\left(M^{n}\setminus\Sigma\right) \text{ where } \Sigma \text{ is} \\ \text{a finite union of } (n-5)\text{-dimensional submanifolds.} \end{array} \right\}$$

Locally, restriction of the bundle to the sphere $\mathbb{S}^4 \subset \mathbb{R}^5$ in the 5-plane transversal to Σ, can have nontrivial second Chern class c_2. But in the smooth case, we can see using **transgression forms** (also called **Chern-Simons forms**)

$$\int_{\mathbb{S}^4} \mathrm{Tr}\left(F_A \wedge F_A\right) = \int_{B_1^5} d\,\mathrm{Tr}\left(F_A \wedge F_A\right) = \int_{B_1^5} d\left[d\,\mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)\right] = 0.$$

Conjecture about density of almost smooth bundles

'Almost' smooth bundles of Petrache-Rivière [10]

$$\mathcal{R}^{\infty}\left(M^n, \mathrm{SU}(2)\right) := \left\{ \begin{array}{c} (P, A) : A \in \mathcal{A}^{\infty}(P)\,,\, P \in \mathcal{P}^{\infty}_{\mathrm{SU}(2)}\left(M^n \setminus \Sigma\right) \text{ where } \Sigma \text{ is} \\ \text{a finite union of } (n-5)\text{-dimensional submanifolds.} \end{array} \right\}$$

Locally, restriction of the bundle to the sphere $\mathbb{S}^4 \subset \mathbb{R}^5$ in the 5-plane transversal to Σ, can have nontrivial second Chern class c_2. But in the smooth case, we can see using **transgression forms** (also called **Chern-Simons forms**)

$$\int_{\mathbb{S}^4} \mathrm{Tr}\left(F_A \wedge F_A\right) = \int_{B_1^5} d\,\mathrm{Tr}\left(F_A \wedge F_A\right) = \int_{B_1^5} d\left[d\,\mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)\right] = 0.$$

In general, $\quad d\,\mathrm{Tr}\left(F_A \wedge F_A\right) = \sum n_i \delta_{x_i} \quad$ in $\mathcal{D}'\left(B_1^5\right)$, is possible.

Conjecture about density of almost smooth bundles

'Almost' smooth bundles of Petrache-Rivière [10]

$$\mathcal{R}^{\infty}\left(M^{n}, \mathrm{SU}(2)\right) := \left\{ \begin{array}{c} (P, A) : A \in \mathcal{A}^{\infty}(P), P \in \mathcal{P}^{\infty}_{\mathrm{SU}(2)}\left(M^{n} \setminus \Sigma\right) \text{ where } \Sigma \text{ is} \\ \text{a finite union of } (n-5)\text{-dimensional submanifolds.} \end{array} \right\}$$

Locally, restriction of the bundle to the sphere $\mathbb{S}^4 \subset \mathbb{R}^5$ in the 5-plane transversal to Σ, can have nontrivial second Chern class c_2. But in the smooth case, we can see using **transgression forms** (also called **Chern-Simons forms**)

$$\int_{\mathbb{S}^4} \mathrm{Tr}\left(F_A \wedge F_A\right) = \int_{B^5_1} d\,\mathrm{Tr}\left(F_A \wedge F_A\right) = \int_{B^5_1} d\left[d\,\mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)\right] = 0.$$

In general, $\quad d\,\mathrm{Tr}\left(F_A \wedge F_A\right) = \sum n_i \delta_{x_i} \quad$ in $\mathcal{D}^{'}\left(B^5_1\right)$, is possible.

Conjecture (ongoing work with Mircea Petrache and Tristan Rivière)

$\mathcal{R}^{\infty}$ is strongly dense in $\mathcal{P}^{1,\mathrm{L}^{4,n-4}}_{\mathrm{SU}(2)} \times \mathcal{U}^{1,\mathrm{L}^{4,n-4}}$.

Preprints and planned works

- The preprint for the article on critical dimension can be found in arXiv [16].
- Supercritical dimension results should appear in arXiv soon, either as a separate article by me, or as part of an article coauthored with Mircea Petrache (PUC Chile) and Tristan Rivière (ETH Zurich), which is still a work in progress.

Acknowledgement

I warmly thank to Tristan Rivière of ETH Zurich for numerous discussions, encouragement and support. He introduced me to this subject. The influence of my discussions with him on these works can not be overstated.

References I

BETHUEL, F.
The approximation problem for Sobolev maps between two manifolds.
Acta Math. 167, 3-4 (1991), 153–206.

BREZIS, H., AND NIRENBERG, L.
Degree theory and BMO. I. Compact manifolds without boundaries.
Selecta Math. (N.S.) 1, 2 (1995), 197–263.

DONALDSON, S. K., AND THOMAS, R. P.
Gauge theory in higher dimensions.
In *The geometric universe (Oxford, 1996)*. Oxford Univ. Press, Oxford, 1998, pp. 31–47.

FREED, D. S., AND UHLENBECK, K. K.
Instantons and four-manifolds, vol. 1 of *Mathematical Sciences Research Institute Publications*.
Springer-Verlag, New York, 1984.

References II

HANG, F., AND LIN, F.
Topology of Sobolev mappings. II.
Acta Math. 191, 1 (2003), 55–107.

ISOBE, T.
Topological and analytical properties of Sobolev bundles. I. The critical case.
Ann. Global Anal. Geom. 35, 3 (2009), 277–337.

ISOBE, T.
Sobolev bundles with abelian structure groups.
Calc. Var. Partial Differential Equations 49, 1-2 (2014), 77–102.

LAWSON, JR., H. B.
The theory of gauge fields in four dimensions, vol. 58 of *CBMS Regional Conference Series in Mathematics*.
Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1985.

References III

MEYER, Y., AND RIVIÈRE, T.
A partial regularity result for a class of stationary Yang-Mills fields in high dimension.
Rev. Mat. Iberoamericana 19, 1 (2003), 195–219.

PETRACHE, M., AND RIVIÈRE, T.
The resolution of the Yang-Mills Plateau problem in super-critical dimensions.
Adv. Math. 316 (2017), 469–540.

PRICE, P.
A monotonicity formula for Yang-Mills fields.
Manuscripta Math. 43, 2-3 (1983), 131–166.

RIVIÈRE, T.
Interpolation spaces and energy quantization for Yang-Mills fields.
Comm. Anal. Geom. 10, 4 (2002), 683–708.

References IV

Schoen, R., and Uhlenbeck, K.
Boundary regularity and the Dirichlet problem for harmonic maps.
J. Differential Geom. 18, 2 (1983), 253–268.

Sedlacek, S.
A direct method for minimizing the Yang-Mills functional over 4-manifolds.
Comm. Math. Phys. 86, 4 (1982), 515–527.

Shevchishin, V. V.
Limit holonomy and extension properties of Sobolev and Yang-Mills bundles.
J. Geom. Anal. 12, 3 (2002), 493–528.

Sil, S.
Topology of weak g-bundles via coulomb gauges in critical dimensions.
arXiv e-prints (September 2019), arXiv:1909.07308.

Tao, T., and Tian, G.
A singularity removal theorem for Yang-Mills fields in higher dimensions.
J. Amer. Math. Soc. 17, 3 (2004), 557–593.

References V

Taubes, C. H.
Metrics, connections and gluing theorems, vol. 89 of *CBMS Regional Conference Series in Mathematics*.
Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1996.

Tian, G.
Gauge theory and calibrated geometry. I.
Ann. of Math. (2) 151, 1 (2000), 193–268.

Uhlenbeck, K. K.
Connections with L^p bounds on curvature.
Comm. Math. Phys. 83, 1 (1982), 31–42.

Thank you

Questions?

Instantons in complex geometry, special Holonomy and calibrated geometry

Self-duality and Yang-Mills fields [3], [19]

A connection form A on an $\mathrm{SU}(r)$ principal bundle over (M^n, g) is Ω-ASD instanton for some closed $(n-4)-$form Ω on M^n if

$$*_g F_A = -\Omega \wedge F_A. \tag{10}$$

- For Ω-ASD instantons, the Bianchi identity implies the YM equation.
- (M^4, g) Riem., $\Omega \equiv 1$. Then (10) $\Leftrightarrow$ ASD instanton.
- (M^{2m}, g) Kähler, $\Omega = \frac{1}{(m-2)!}\omega_g^{m-2}$. Then (10) $\Leftrightarrow$ Hermitian-YM equation.
- (M^8, g) is a Calabi-Yau 4-fold, θ is holomorphic $(4,0)$ form with $\theta \wedge \bar{\theta} = \frac{1}{4!}\omega_g^4$. Take $\Omega = 4\,\mathrm{Re}(\theta) + \frac{1}{2}\omega_g^2$. Then (10) $\Leftrightarrow$ $\mathrm{SU}(4)$-instanton equation.
- (M^8, g) is a $\mathrm{Spin}(7)$ manifold. There is a parallel 4-form Ω, left invariant by the action of $\mathrm{Spin}(7)$ such that (10) becomes $\mathrm{Spin}(7)$-instanton equation.
- (M^7, g) is a G_2 manifold. There is a parallel 3-form Ω left invariant by the action of G_2 such that (10) is called G_2-instanton equation.