Gauge freedom in variational problems for differential forms

Swarnendu Sil

Forschungsinstitut für Mathematik
ETH Zürich

22nd May 2020
Research Summary Presentation
Selection Committee Online Meeting
Indian Institute of Science
Bangalore, India

Differential forms and gauge freedom

The central theme

Analysis of variational problems involving differential forms and natural differential operators on forms and the related system of PDEs.

The key feature: Gauge invariance

Large, typically infinite dimensional, group of symmetries. Related system of PDEs are only elliptic modulo the action of an infinite dimensional invariance group.

My work can be divided in two somewhat distinct regimes.

- **Vector-valued forms on Euclidean domains** Features like an Abelian gauge theory. The question is **up to what extent classical results survive the gauge freedom**.
- **Connection and curvature forms on principal bundles** This is gauge theory proper. **Geometry and topology interacts strongly with analysis**. Features typical of geometric analysis problems, e.g. Harmonic maps.

Functional depending on exterior derivative of a differential form

$n \geq 2$, $N \geq 1$, $0 \leq k \leq n-1$ are integers. $\Omega \subset \mathbb{R}^n$ open, bounded, smooth.
Consider vector-valued k-forms $u : \Omega \to \Lambda^k \mathbb{R}^n \otimes \mathbb{R}^N$.

Existence question for differential k-forms

Given $f : \Lambda^{k+1}\mathbb{R}^n \otimes \mathbb{R}^N \to \mathbb{R}$, show the existence of a minimizer for

$$m := \min \left\{ I(u) := \int_\Omega f\left(du\right) : u \in u_0 + W_0^{1,p}\left(\Omega; \Lambda^k \mathbb{R}^n \otimes \mathbb{R}^N\right) \right\} \tag{1}$$

Here $1 < p < \infty$. Typically, one assumes p-growth and p-coercivity, i.e.

$$c_1\left(|\xi|^p - 1\right) \leq f(\xi) \leq c_2\left(|\xi|^p + 1\right) \quad \text{for every } \xi \in \Lambda^k \mathbb{R}^n \otimes \mathbb{R}^N. \tag{2}$$

The gauge freedom for $1 \leq k \leq n-1$

As soon as $k \neq 0$, then we have $I(u) = I(u+v)$ for any closed form $v \in W_0^{1,p}$.

Convexity in the vectorial calculus of variations

The case $k = 0$ is studied extensively in the calculus of variations, where $N = 1$ is the scalar case and $N \geq 2$ is the vectorial case. (see Dacorogna's book [4]). For $k = 0$, (2) implies that minimizing sequences u_n satisfy $\|\nabla u_n\|_{L^p} \leq C$. So we only need $I(\cdot)$ to be sequentially weakly lower semicontinuous, i.e.

$$I(u) \leq \liminf I(u_n) \quad \text{for every } u_n \overset{W^{1,p}}{\rightharpoonup} u.$$

Sequential weak lower semicontinuity of I depends on convexity properties of f. In the vectorial case, one has notions of **polyconvexity**, **quasiconvexity** and **rank one convexity**.

Existence of minimizers can be proved for quasiconvex integrands. But quasiconvexity is harder to check. In practice, polyconvexity is most useful.

Relations of different notions of convexity

Convex $\Rightarrow$ polyconvex $\Rightarrow$ quasiconvex $\Rightarrow$ rank one convex. In each case, converse implication is false in general. Their interrelations have been extensively studied (see [4]). Polyaffine $\Leftrightarrow$ quasiaffine $\Leftrightarrow$ rank one affine, but not necessarily affine.

Quasiaffine maps and weak continuity

The notion of polyconvexity is intimately related to the question

Characterization of weak continuity

Characterize all $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ that satisfies (assuming growth assumptions)

$$f(\nabla u_n) \stackrel{\mathcal{D}'}{\rightharpoonup} f(\nabla u) \quad \text{for all } u_n \stackrel{W^{1,p}}{\rightharpoonup} u, \tag{3}$$

Theorem (Ball, ARMA 76/77 [1])

(3) *holds if and only if, for some constant vectors* c_s, *we have*

$$f(\xi) = \sum_{s=0}^{\min\{n,N\}} \langle c_s, \operatorname{adj}_s(\xi) \rangle .$$

Here $\operatorname{adj}_s(\xi)$ is the vector with determinants of $s \times s$ minors of ξ as components.

The case $N = 1$

In my PhD work, I started investigating, partly in collaboration with my advisor Bernard Dacorogna (EPFL) and Saugata Bandyopadhyay (IISER Kolkata), whether one can have a framework for direct methods in the cases with $1 \leq k \leq n - 1$ as well.

$N = 1$ case: Bandyopadhyay-Dacorogna-S., JEMS, 2016 [2]

- Developed the framework for the case $N = 1$ and showed one can have an analogous, parallel but distinct theory in this case as well.
- Introduced the notion of **ext. polyconvexity**, **ext. quasiconvexity** and **ext. one convexity** and provided a near complete study of their interrelations.
- Established the existence of minimizers for ext. quasiconvex integrands.

The general case $N \geq 1$

Existence results were established for $N \geq 1$ case in **S.**, Adv. Calc. Var, 2019 [8], by introducing the notions of **vectorial ext. polyconvexity**, **vectorial ext. quasiconvexity** and **vectorial ext. one convexity**. It also showed the following

Theorem (S., Adv. Calc. Var, 2019 [8])

$f : \Lambda^{k+1}\mathbb{R}^n \otimes \mathbb{R}^N \to \mathbb{R}$ *satisfies (assuming growth assumptions),*

$f(du_n) \overset{\mathcal{D}'}{\rightharpoonup} f(du)$ *for all* $u_n \overset{W^{1,p}}{\rightharpoonup} u$, **if and only if** *$f$ is of the form*

$$f(\xi) = \sum_{0 \leq |\alpha| \leq n} \langle c_\alpha, \xi^\alpha \rangle$$

where $\alpha = (\alpha_1, \ldots, \alpha_N)$ *is a multiindex,* $|\alpha| = \sum_{i=1}^N \alpha_i$, c_α*s are constant forms and* $\xi^\alpha = \xi_1^{\alpha_1} \wedge \ldots \wedge \xi_N^{\alpha_N}$ *are wedge products of wedge powers.*

This gives a **new proof** of Ball's theorem and in a sense also **explains why adjugates appear** there.

Gauge fixing trick for existence

For a minimizing sequence u_n, (2) implies only $\|du_n\|_{L^p} \leq C$, but not $\|\nabla u_n\|_{L^p} \leq C$. The trick is use **gauge fixing**. For each n, we solve,

$$\begin{cases} dv_n = du_n & \text{in } \Omega, \\ d^* v_n = 0 & \text{in } \Omega, \\ \iota^*_{\partial\Omega} v_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

This would imply, $\|v_n\|_{W^{1,p}} \leq C \|du_n\|_{L^p} \leq C$ and thus $v_n \overset{W^{1,p}}{\rightharpoonup} v$. Now we solve

$$\begin{cases} du = dv & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Thus, by sequential weak lower semicontinuity, we have,

$$m \leq I(u) \overset{(5)}{=} I(v) \leq \liminf I(v_n) \overset{(4)}{=} \liminf I(u_n) = m.$$

Quasilinear systems

Quasilinear systems for forms

Consider the following system for a vector-valued form $u : \Omega \to \Lambda^k \mathbb{R}^n \otimes \mathbb{R}^N$,

$$d^* \left(a(x) \left| du \right|^{p-2} du \right) = f \qquad \text{in } \Omega. \tag{6}$$

Can we derive **regularity estimates for du in terms of f?**

- For $1 \leq k \leq n-1$, for any solution u and any closed form v, $u + v$ is also a solution. Also, $d^* f = 0$ is a necessary condition for the existence of a solution.
- If $f = 0$ and a is a constant function, then a fundamental estimate due to Uhlenbeck [12] tells us that u is locally $C^{1,\alpha}$ for some $0 < \alpha < 1$.
- For nonzero f, known results are only for $k = 0$, which have been studied extensively. For $a \equiv 1$, the case $k = 0, N = 1$ is the p-Laplacian equation and $k = 0, N \geq 2$ is the p-Laplacian system.

A nonlinear Stein theorem

Stein [11] established the borderline case of Sobolev embedding

$$\nabla u \in L^{(n,1)}_{loc} \Rightarrow u \text{ is continuous.}$$

Combined with Calderon-Zygmund estimates for the Laplacian in Lorentz spaces, we get

$$\Delta u \in L^{(n,1)}_{loc} \Rightarrow \nabla u \text{ is continuous.}$$

The following is a nonlinear version for forms.

Theorem (S., Calc. Var. PDE 2019 [9])

If $a : \Omega \to [\gamma, L]$ is Dini continuous with $0 < \gamma < L < \infty$ and $f \in L^{(n,1)}_{loc}\left(\Omega; \Lambda^k \mathbb{R}^n \otimes \mathbb{R}^N\right)$, then du is continuous in Ω for any u solving (6).

The case $k = 0$, where there is no gauge freedom, was proved in Kuusi-Mingione 2014 [5].

New ingredients

The following new Poincaré-Sobolev type inequality is crucial.

Proposition (S., Calc. Var. PDE 2019 [9])

For any $1 < p < n$, *there is a constant* $c > 0$ *such that any* $u \in W^{1,p}\left(B_R; \Lambda^k \mathbb{R}^n \otimes \mathbb{R}^N\right)$ *with* $d^* u = 0$ *in* B_R *and* $\iota^*_{\partial B_R} u = 0$ *on* ∂B_R, *we have*

$$\left\| u \right\|_{L^{\frac{np}{n-p}}(B_R)} \leq c \left\| du \right\|_{L^p(B_R)} .$$

For obtaining comparison estimates, one needs to use systems like

$$\begin{cases} d^* \left(a(x) \left| dw \right|^{p-2} dw \right) = 0 & \text{in } B_R(x_0) \\ d^* w = d^* u & \text{in } B_R(x_0) \\ \iota^*_{\partial B_R(x_0)} w = \iota^*_{\partial B_R(x_0)} u & \text{on } \partial B_R(x_0). \end{cases}$$

One also needs to work with the gauge fixed system

$$d^* \left(a(x) \left| du \right|^{p-2} du \right) = f \quad \text{and } d^* u = 0 \qquad \text{in } \Omega.$$

The constant in Gaffney inequality

The effectiveness of Coulomb gauges is due to the following important inequality.

Gaffney inequality

$\Omega \subset \mathbb{R}^n$ open, bdd, smooth; $u \in W^{1,2}\left(\Omega; \Lambda^k \mathbb{R}^n\right)$ with $\iota^*_{\partial\Omega} u = 0$ on $\partial\Omega$, then

$$\|\nabla u\|_{L^2} \leq C_{\Omega,k}\left(\|du\|_{L^2} + \|d^* u\|_{L^2} + \|u\|_{L^2}\right).$$

We can show that $C_{\Omega,k} \geq 1$ for any Ω and any $1 \leq k \leq n-1$. ($C_{\Omega,0} = 1 = C_{\Omega,n}$). Also, we can characterize the constant by

$$C_{\Omega,k} = \sup_{u \in W^{1,2}_T(\Omega;\Lambda^k\mathbb{R}^n)\setminus\{0\}} \frac{\|\nabla u\|_{L^2}}{\left(\|du\|_{L^2} + \|d^* u\|_{L^2} + \|u\|_{L^2}\right)}. \tag{7}$$

Characterizing the best cases

- Can we characterize all smooth domains Ω with $C_{\Omega,k} = 1$?
- Is the supremum in (7) attained?

Domains with the best Gaffney constant

It was known from Mitrea [6] that Ω is $(n-k)$-convex $\Rightarrow C_{\Omega,k} = 1$.
In collaboration with Gyula Csató (Universitat de Barcelona) and Bernard Dacorogna (EPFL), I was able to answer this question completely.

Theorem (Csato-Dacorogna-S, JFA 2018 [3])

The following are equivalent.

- **(i)** $C_{\Omega,k} = 1$.
- **(ii)** Ω *is* $(n-k)$*-convex.*
- **(iii)** *The supremum in* (7) *is not attained.*
- **(iv)** *For all* $u \in W^{1,2}\left(\Omega; \Lambda^k \mathbb{R}^n\right)$ *with* $\iota^*_{\partial\Omega} u = 0$ *on* $\partial\Omega$*, we have*

$$\|\nabla u\|_{L^2} \leq \|du\|_{L^2} + \|d^* u\|_{L^2}.$$

- **(v)** $C_{\Omega,k}$ *is scale-invariant, i.e. for any* $t > 0$*, we have* $C_{t\Omega,k} = C_{\Omega,k}$.

Note that (iv) implies $H^k_{DeRham}(\Omega) = \{0\}$, a vanishing theorem.

Gauge theory in the critical dimension

Local picture

$A \in W^{1,2}\left(B^4; \Lambda^1\mathbb{R}^4 \otimes \mathfrak{su}(2)\right)$ be a connection form on $B^4 \times \mathrm{SU}(2)$, the trivial $\mathrm{SU}(2)$ principal bundle over the unit ball $B^4 \subset \mathbb{R}^4$.
The curvature is

$$F_A = dA + A \wedge A \in L^2\left(B^4; \Lambda^2\mathbb{R}^4 \otimes \mathfrak{su}(2)\right).$$

Given a gauge $\rho : B^4 \to \mathrm{SU}(2)$, the connection in this new gauge is

$$A^\rho = \rho^{-1} d\rho + \rho^{-1} A \rho.$$

The curvature becomes

$$F_{A^\rho} = \rho^{-1} F_A \rho.$$

So we have $YM(A) = YM(A^\rho)$, where $YM(\cdot)$ is the Yang-Mills energy,

$$YM(A) := \int_{B^4} |F_A|^2 := -\int_{B^4} \mathrm{Tr}\left(F_A \wedge *F_A\right).$$

Gauge fixing

Coulomb gauges

The curvature F_A has a term containing dA, so one can try to obtain $d^*A = 0$. Even if $d^*A = 0$, it is still not clear if $W^{1,2}$ norm of A can be controlled by the L^2 norm of F_A due to the nonlinear term $A \wedge A$.

Theorem (Uhlenbeck, Comm. Math. Phys., 1982 [13])

There exists an absolute constant $\varepsilon_{Uh} > 0$ *such that if*

$$\int_{B^4} |F_A|^2 < \varepsilon_{Uh},$$

then there exists a Coulomb gauge $\rho \in W^{2,2}\left(B^4; \mathrm{SU}(2)\right)$ *such that*

$$d^*A^\rho = 0 \quad \text{in } B^4 \ \textbf{(Coulomb condition)} \ , \qquad \iota^*_{\partial B^4}\left(*A^\rho\right) = 0 \text{ on } \partial B^4,$$

and

$$\int_{B^4} |\nabla A^\rho|^2 + \int_{B^4} |A^\rho|^4 \leq C \int_{B^4} |F_A|^2 .$$

Gauge theory on $\mathbb{S}^4$

$A \in W^{1,2}\left(\mathbb{S}^4; \Lambda^1 T^*\mathbb{S}^4 \otimes \mathfrak{su}(2)\right)$ is a connection form on $\mathbb{S}^4 \times \mathrm{SU}(2)$.
Pick a cover $\cup_i U_i = \mathbb{S}^4$ such that

$$\int_{U_i} |F_A|^2 < \varepsilon_{Uh} \qquad \text{for each } i.$$

Then for each i, we obtain maps $\rho_i \in W^{2,2}\left(U_i; \mathrm{SU}(2)\right)$, which is a local Coulomb gauge for $A_i := A|_{U_i}$, i.e.

$$d^*_{\mathbb{S}^4} A_i^{\rho_i} = 0 \qquad \text{in } U_i.$$

The maps

$$g_{ij} := \rho_i^{-1}\rho_j \in W^{2,2}\left(U_i \cap U_j; \mathrm{SU}(2)\right)$$

define a $W^{2,2}$ bundle P over $\mathbb{S}^4$ and the local connection forms

$$B_i := A_i^{\rho_i} \in W^{1,2}\left(U_i; \Lambda^1 T^* U_i \otimes \mathfrak{su}(2)\right)$$

defines a $W^{1,2}$ connection form on P which is Coulomb.

Regularity of Coulomb bundles

Since $W^{2,2} \not\hookrightarrow C^0$ in dimension 4, P is not a priori a C^0 bundle. But Rivière [7] showed in 2002 that bundles admitting a $W^{1,2}$ Coulomb connection are C^0 bundles, using improved Sobolev embedding into Lorentz spaces.

Recently, I managed to show a stronger regularity result via a different method.

Theorem (S., Preprint, 2019 [10])

If A is an $W^{1,2}$ connection on a $W^{2,2}$ principal $\mathrm{SU}(2)$-bundle P over $\mathbb{S}^4$ such that A is Coulomb, then P is a $W^{2,p} \cap C^{0,\alpha}$ bundle, for any $0 < \alpha < 1$ and any $p < 4$.

The regularity result strongly uses that the connection is Coulomb. For a $W^{2,2}$ bundles P over $\mathbb{S}^4$ with a general $W^{1,2}$ connection A on it, one can ask

Smooth approximation and topological isomorphism class

- Can we find **smooth** $(P^{\nu}, A^{\nu}) \to (P, A)$ in the respective **strong topologies**?
- Can we associate, in a meaningful and useful way, a **topological isomorphism class** to (P, A)?

Approximation and topology for $W^{2,2}$ bundles with $W^{1,2}$ connections

Theorem (S., Preprint, 2019 [10])

Given any $W^{2,2}$ principal $\mathrm{SU}(2)$-bundle P over $\mathbb{S}^4$ and a $W^{1,2}$ connection A on P, there exists a sequence of smooth principal $\mathrm{SU}(2)$-bundles P^ν with smooth connections A^ν on P^ν such that $P^\nu \overset{W^{2,2}}{\simeq}_{\rho^\nu} P$ and locally, $g^\nu_{ij} \to g_{ij}$ in $W^{2,2}$ and $A^\nu_i - (\rho^\nu_i)^* A_i \to 0$ in $W^{1,2}$, where g^ν_{ij}, g_{ij} are the respective transition functions.

Such a pair (P, A) can also be assigned a topological isomorphism class.

Theorem (S., Preprint, 2019 [10])

Given any $W^{2,2}$ principal $\mathrm{SU}(2)$-bundle P over $\mathbb{S}^4$ and a $W^{1,2}$ connection A on P, there exists a C^0 bundle $\bar{P}$, unique up to topological isomorphism, such that $P \overset{W^{2,2}}{\simeq} \bar{P}$. This assignment is stable under $W^{2,2}$ gauge transformations of (P, A).

This assignment is for the pair (P, A), not the bundle P alone. Two topologically distinct bundle can be gauge related via $W^{2,2}$ gauges.

References I

BALL, J. M.
Convexity conditions and existence theorems in nonlinear elasticity.
Arch. Rational Mech. Anal. 63, 4 (1976/77), 337–403.

BANDYOPADHYAY, S., DACOROGNA, B., AND SIL, S.
Calculus of variations with differential forms.
J. Eur. Math. Soc. (JEMS) 17, 4 (2015), 1009–1039.

CSATO, G., DACOROGNA, B., AND SIL, S.
On the best constant in Gaffney inequality.
J. Funct. Anal. 274, 2 (2018), 461–503.

DACOROGNA, B.
Direct methods in the calculus of variations, second ed., vol. 78 of *Applied Mathematical Sciences*.
Springer, New York, 2008.

References II

KUUSI, T., AND MINGIONE, G.
A nonlinear Stein theorem.
Calc. Var. Partial Differential Equations 51, 1-2 (2014), 45–86.

MITREA, M.
Dirichlet integrals and Gaffney-Friedrichs inequalities in convex domains.
Forum Math. 13, 4 (2001), 531–567.

RIVIÈRE, T.
Interpolation spaces and energy quantization for Yang-Mills fields.
Comm. Anal. Geom. 10, 4 (2002), 683–708.

SIL, S.
Calculus of variations: A differential form approach.
Adv. Calc. Var. 12, 1 (2019), 57–84.

SIL, S.
Nonlinear Stein theorem for differential forms.
Calc. Var. Partial Differential Equations 58, 4 (2019), Paper No. 154.

References III

Sil, S.
Topology of weak G-bundles via Coulomb gauges in critical dimensions.
arXiv e-prints (September 2019), arXiv:1909.07308.

Stein, E. M.
Editor's note: the differentiability of functions in $\mathbf{R}^n$.
Ann. of Math. (2) 113, 2 (1981), 383–385.

Uhlenbeck, K.
Regularity for a class of non-linear elliptic systems.
Acta Math. 138, 3-4 (1977), 219–240.

Uhlenbeck, K. K.
Connections with L^p bounds on curvature.
Comm. Math. Phys. 83, 1 (1982), 31–42.

Thank you

Questions?