Detailed proof of the Characterization theorem for Quasiaffine maps of several differential forms

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The purpose of this note is to spell out in detail a piece of argument employed in the proof of the characterization theorem for the quasiaffine functions for calculus of variations for several differential forms (Theorem 2 here).

The theorem was first obtained in my PhD Thesis (cf. Theorem 4.12 in [4]) and also appeared as Theorem 3.9 in [3]. Unfortunately, this last piece of argument was not spelled out explicitly enough in the published versions of either of them. Although it is basically an iteration of the exact same argument employed in the proof of Theorem 3.3 in [2], still at present I think (contrary to my younger self) the details are not obvious enough to warrant skipping. Here it is written in a self-contained manner.

Convexity notions and basic Properties

The definitions of different notions of vectorial ext. convexity and affinity can be found in [3], [4]. We recall their basic relationships.

Theorem 1 *Let $f : \mathbf{\Lambda}^{\boldsymbol{k}} \to \mathbb{R}$. Then*

$$\begin{aligned} f\,\text{convex} \Rightarrow f \text{ vectorially ext. polyconvex} &\Rightarrow f \text{ vectorially ext. quasiconvex} \\ &\Rightarrow f \text{ vectorially ext. one convex.} \end{aligned}$$

Moreover if $f : \mathbf{\Lambda}^{\boldsymbol{k}}(\mathbb{R}^n) \to \mathbb{R}$ is vectorially ext. one convex, then f is locally Lipschitz.

Proof can be found in [3], [4].

The quasiaffine case

We now prove the basic characterization theorem for vectorially ext. quasiaffine functions. In the special case when $k_i = 1$ for all $1 \leq i \leq m$, this immediately implies classical theorem of Ball [1] with a new proof. In a sense, this theorem also 'explains' the appearance of determinants and adjugates in the classical theorem. Determinants and adjugates appear as they are precisely the 'wedge products' in the classical case.

Theorem 2 *Let $f : \mathbf{\Lambda}^{\boldsymbol{k}} \to \mathbb{R}$. The following statements are then equivalent.*

(i) *f is vectorially ext. polyaffine.*

(ii) *f is vectorially ext. quasiaffine.*

(iii) *f is vectorially ext. one affine.*

(iv) *There exist $c_{\boldsymbol{\alpha}} \in \Lambda^{|\boldsymbol{k\alpha}|}(\mathbb{R}^n)$, for every $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$ such that $0 \leq \alpha_i \leq \left[\frac{n}{k_i}\right]$ for all $1 \leq i \leq m$ and $0 \leq |\boldsymbol{k\alpha}| \leq n$, such that for every $\boldsymbol{\xi} \in \mathbf{\Lambda}^{\boldsymbol{k}}$,*

$$f(\boldsymbol{\xi}) = \sum_{\substack{\boldsymbol{\alpha}, \\ 0 \leq |\boldsymbol{k\alpha}| \leq n}} \langle c_{\boldsymbol{\alpha}}; \boldsymbol{\xi}^{\boldsymbol{\alpha}} \rangle .$$

Proof $(i) \Rightarrow (ii) \Rightarrow (iii)$ follows from Theorem 1. $(iv) \Rightarrow (i)$ is immediate from the definition of vectorial ext. polyconvexity. So we only need to show $(iii) \Rightarrow (iv)$.

We show this by induction on m. Clearly, for $m = 1$, this is just the characterization theorem for ext. one affine functions, given in theorem 3.3 in [2]. We assume the result to be true for $m \leq p-1$ and show it for $m = p$. Now since f is vectorially ext. one affine, it is separately ext. one affine and using ext. one affinity with respect to ξ_p, keeping the other variables fixed, we obtain,

$$f(\boldsymbol{\xi}) = \sum_{s=1}^{[\frac{n}{k_p}]} \langle c_s(\xi_1, \ldots, \xi_{p-1}); \xi_p^s \rangle,$$

where for each $1 \leq s \leq [\frac{n}{k_p}]$, the functions $c_s : \prod_{i=1}^{p-1} \Lambda^{k_i} \to \Lambda^{sk_p}$ are such that the map $(\xi_1, \ldots, \xi_{p-1}) \mapsto f(\xi_1, \ldots, \xi_{p-1}, \xi_p)$ is vectorially ext. one affine for any $\xi_p \in \Lambda^{k_p}$. Arguing by degree of homogeneity, this implies that for each $1 \leq s \leq [\frac{n}{k_p}]$, every component c_S^I is vectorially ext. one affine, i.e $(\xi_1, \ldots, \xi_{p-1}) \mapsto c_s^I(\xi_1, \ldots, \xi_{p-1})$ is vectorially ext. one affine for any $I \in \mathcal{T}^{sk_p}$. Applying the induction hypothesis to each of these components, multiplying out and collecting terms of according to the degree of homogeneity, we obtain,

$$f(\boldsymbol{\xi}) = \sum_{\substack{\boldsymbol{\alpha}, \\ 0 \leq |\boldsymbol{k\alpha}| \leq n}} f_{\boldsymbol{\alpha}}(\boldsymbol{\xi}),$$

where

$$f_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \sum_{I \in \mathcal{T}^{k_p \alpha_p}} \left\langle c_{\boldsymbol{\alpha}, I}; \xi_1^{\alpha_1} \wedge \ldots \wedge \xi_{p-1}^{\alpha_{p-1}} \right\rangle \left(\xi_p^{\alpha_p}\right)^I,$$

and $c_{\boldsymbol{\alpha},I} \in \mathcal{T}^{(k_1\alpha_1+\ldots+k_{p-1}\alpha_{p-1})}$ is a constant form for each $I \in \mathcal{T}^{k_p\alpha_p}$.

Now we just need to show that $f_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \langle c_{\boldsymbol{\alpha}}; \boldsymbol{\xi}^{\boldsymbol{\alpha}} \rangle$, for some constant form $c_{\boldsymbol{\alpha}} \in \Lambda^{|\boldsymbol{k}\boldsymbol{\alpha}|}(\mathbb{R}^n)$. This is done by the same arguments that were used in the proof of Theorem 3.3 in [2]. Again by a degree of homogeneity argument (just notice that the map $t \mapsto f_\alpha(\lambda_1\xi_1 + t\lambda_1 a \wedge b_1, \ldots, \lambda_p\xi_p + t\lambda_p a \wedge b_p)$ must be affine in t for all choices of $\lambda_1, \ldots, \lambda_p \in \mathbb{R}$) , it is clear that each f_α must be vectorially ext. quasiaffine themselves. Thus it is enough to restrict our attention to a fixed, but arbitrary admissible $\boldsymbol{\alpha}$. Note that we can write

$$
\begin{aligned}
&\langle c_\alpha; \boldsymbol{\xi}^{\boldsymbol{\alpha}} \rangle \\
&= \left\langle \xi_p^{\alpha_p}; \left(\xi_1^{\alpha_1} \wedge \ldots \wedge \xi_{p-1}^{\alpha_{p-1}}\right) \lrcorner c_{\boldsymbol{\alpha}} \right\rangle \\
&= \sum_{I \in \mathcal{T}^{k_p\alpha_p}} \left(\xi_p^{\alpha_p}\right)^I \left(\left(\xi_1^{\alpha_1} \wedge \ldots \wedge \xi_{p-1}^{\alpha_{p-1}}\right) \lrcorner c_{\boldsymbol{\alpha}}\right)^I \\
&= \sum_{I \in \mathcal{T}^{k_p\alpha_p}} \left(\xi_p^{\alpha_p}\right)^I \sum_{J \in \mathcal{T}^{\left(\sum_{i=1}^{p-1} k_i\alpha_i\right)}} \left(\xi_1^{\alpha_1} \wedge \ldots \wedge \xi_{p-1}^{\alpha_{p-1}}\right)^J c_{\boldsymbol{\alpha}}^{[IJ]} (-1)^{k_p\alpha_p} \operatorname{sgn}[I,J]. \quad (1)
\end{aligned}
$$

Finding the strange looking formula for the sign is rather easy, illustrating how efficient our notations really are. Indeed, the correct sign the obviously the sign needed to make the expression $\left(e^J \lrcorner e^{[IJ]}\right)$ equal to e^I, i.e the sign of the expression $\left\langle e^J \lrcorner e^{[IJ]}; e^I \right\rangle$. But this is equal to

$$
\begin{aligned}
\left\langle e^J \lrcorner e^{[IJ]}; e^I \right\rangle &= (-1)^{k_p\alpha_p} \left\langle e^{[IJ]}; e^I \wedge e^J \right\rangle \qquad (\text{since } I \in \mathcal{T}^{k_p\alpha_p}), \\
&= (-1)^{k_p\alpha_p} \left\langle e^{[IJ]}; \operatorname{sgn}[I,J] e^{[IJ]} \right\rangle = (-1)^{k_p\alpha_p} \operatorname{sgn}[I,J].
\end{aligned}
$$

On the other hand, we have,

$$
f_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \sum_{I \in \mathcal{T}^{k_p\alpha_p}} \left(\xi_p^{\alpha_p}\right)^I \sum_{J \in \mathcal{T}^{\left(\sum_{i=1}^{p-1} k_i\alpha_i\right)}} \left(\xi_1^{\alpha_1} \wedge \ldots \wedge \xi_{p-1}^{\alpha_{p-1}}\right)^J \left(c_{\boldsymbol{\alpha},I}\right)^J. \qquad (2)
$$

So from (1) and (2), we see that to prove the theorem all we need to show is that

$$
\left(c_{\boldsymbol{\alpha},I}\right)^J = 0, \qquad (3)
$$

whenever I and J has an index in common and

$$
\operatorname{sgn}[I,J] \left(c_{\boldsymbol{\alpha},I}\right)^J = \operatorname{sgn}[\widetilde{I},\widetilde{J}] \left(c_{\boldsymbol{\alpha},\widetilde{I}}\right)^{\widetilde{J}}, \qquad (4)
$$

for any choice of $I, \widetilde{I} \in \mathcal{T}^{k_p\alpha_p}$, $J, \widetilde{J} \in \mathcal{T}^{\left(\sum_{i=1}^{p-1} k_i\alpha_i\right)}$ such that $I \cap J = \emptyset = \widetilde{I} \cap \widetilde{J}$ and $I \cup J = \widetilde{I} \cup \widetilde{J}$.

Now we show (3). Fix I, J such that i is a common index between I and J. Since $I \in \mathcal{T}^{k_p\alpha_p}$, we split I into α_p number of multiindices $I^1, \ldots, I^{\alpha_p}$, each

containing k_p indices in a such a way such that $i \in I^1$. Now choose $\xi_p = \sum_{l=2}^{\alpha_p} e^{I_l}$.
Note that this implies the only term containing t in

$$\left(\xi_p + te^i \wedge \operatorname{sgn}[i, I^1_{\widehat{i}}]e^{I^1_{\widehat{i}}}\right)^{\alpha_p} \quad \text{is} \quad te^I,$$

upto multiplication by a positive constant. Similarly since $J \in \mathcal{T}^{\left(\sum_{i=1}^{p-1} k_i\alpha_i\right)}$, we split J into $p-1$ blocks $J_1, \ldots, J_{p-1}$ such that J_l containes $k_l\alpha_l$ indices for each $1 \leq l \leq p-1$. Once again, suppose $i \in J_1$. Now for each $1 \leq l \leq p-1$, we split J_l into α_l multiindices $J_l^1, \ldots, J_l^{\alpha_l}$, each containing k_l indices and assume that $i \in J_1^1$. Now we chose

$$\xi_l = \sum_{r=1}^{\alpha_l} e^{J_l^r} \quad \text{for } 2 \leq l \leq p-1 \quad \text{and } \xi_1 = \sum_{r=2}^{\alpha_1} e^{J_1^r}.$$

Note that this implies the only term containing t in

$$\left(\xi_1 + te^i \wedge \operatorname{sgn}[i, \left(J_1^1\right)_{\widehat{i}}]e^{\left(J_1^1\right)_{\widehat{i}}}\right)^{\alpha_1} = \left(\xi_1 + te^{J_1^1}\right)^{\alpha_1} \quad \text{is} \quad te^{J^1},$$

upto multiplication by a positive constant. Thus the only term quadratic in t in the expression of

$$f_{\boldsymbol{\alpha}}\left(\xi_1 + te^i \wedge \operatorname{sgn}[i, \left(J_1^1\right)_{\widehat{i}}]e^{\left(J_1^1\right)_{\widehat{i}}}, \xi_2, \ldots, \xi_{p-1}, \xi_p + te^i \wedge \operatorname{sgn}[i, I^1_{\widehat{i}}]e^{I^1_{\widehat{i}}}\right)$$

is

$$t^2 \left(c_{\boldsymbol{\alpha},I}\right)^J, \text{ upto multiplication by a positive constant.}$$

Since $f_{\boldsymbol{\alpha}}$ is vectorially ext. one affine, this proves (3).

Now fix I, J such that $I \cap J = \emptyset$ and show (4). Since the permutation of the indices mapping the string of indices (I, J) to the string $(\widetilde{I}, \widetilde{J})$ can always be written as a product of 1-flips , we see it is enough to prove (4) for a 1-flip (see Notations) i.e to show

$$\left(c_{\boldsymbol{\alpha},I}\right)^J = -\left(c_{\boldsymbol{\alpha},\widetilde{I}}\right)^{\widetilde{J}}, \tag{5}$$

when $\widetilde{I}, \widetilde{J}$ is obtained from I, J by a 1-flip, i.e shifting one index of I to J and one index of J to I.

We now show (5) by using the fact that $f_{\boldsymbol{\alpha}}$ is vectorially ext. quasiaffine. Let $i \in I$ and $j \in J$ be the indices that has been flipped to obtain $\widetilde{I}$ and $\widetilde{J}$. In our notations, $\widetilde{I} = [jI_{\widehat{i}}]$ and $\widetilde{J} = [iJ_{\widehat{j}}]$. Once again, we split I into α_p number of multiindices $I^1, \ldots, I^{\alpha_p}$, each containing k_p indices in a such a way such that $i \in I^1$. Also, we split J into into $p-1$ blocks $J_1, \ldots, J_{p-1}$ such that J_l containes $k_l\alpha_l$ indices for each $1 \leq l \leq p-1$, such that $j \in J_1$ and then for each $1 \leq l \leq p-1$, we split J_l into α_l multiindices $J_l^1, \ldots, J_l^{\alpha_l}$, each containing k_l indices such that we have $j \in J_1^1$. Now we chose

$$\xi_1 = \sum_{r=2}^{\alpha_1} e^{J_1^r}, \quad \xi_p = \sum_{l=2}^{\alpha_p} e^{I_l}$$
$$\text{and } \xi_l = \sum_{r=1}^{\alpha_l} e^{J_l^r} \qquad \text{for } 2 \leq l \leq p-1.$$

Now if we chose 1-forms $a = e^i, b = e^j$, $(k_p - 1)$-forms $A_p = \operatorname{sgn}[i, I_{\widehat{i}}^1] e^{I_{\widehat{i}}^1}$, $B_p = 0$, $(k_1 - 1)$-forms $A_1 = 0$ and $B_1 = \operatorname{sgn}[j, \left(J_1^1\right)_{\widehat{j}}] e^{\left(J_1^1\right)_{\widehat{j}}}$, then it is easy to see that we have

$$\left(c_{\boldsymbol{\alpha}, I}\right)^J = f_{\boldsymbol{\alpha}}\left(\xi_1 + a \wedge A_1 + b \wedge B_1, \xi_2, \ldots, \xi_{p-1}, \xi_p + a \wedge A_p + b \wedge B_p\right)$$

and

$$\left(c_{\boldsymbol{\alpha}, \widetilde{I}}\right)^{\widetilde{J}} = f_{\boldsymbol{\alpha}}\left(\xi_1 + a \wedge B_1 + b \wedge A_1, \xi_2, \ldots, \xi_{p-1}, \xi_p + a \wedge B_p + b \wedge A_p\right).$$

Choosing $A_i, B_i = 0 \in \mathcal{T}^{k_i - 1}$ for all $2 \leq i \leq p-1$ and using Lemma 3 and noting the fact that all other terms of (6) are zero, we immediately obtain the claim. ■

The following lemma is an analogue of Corollary 3.2 in [2] (see also Lemma 3.17 and Corollary 3.18 in [4]) and is proved in the same way.

Lemma 3 *Let $f : \boldsymbol{\Lambda^k} \to \mathbb{R}$ be vectorially ext. one affine. Then for every $a, b \in \Lambda^1$ and every collection of $A_i, B_i \in \mathcal{T}^{k_i - 1}$, $1 \leq i \leq m$ and every $\boldsymbol{\xi} \in \boldsymbol{\Lambda^k}$, we have*

$$\begin{aligned}
&\left[f\left(\xi_1 + a \wedge A_1 + b \wedge B_1, \ldots, \xi_m + a \wedge A_m + b \wedge B_m\right) - f(\boldsymbol{\xi})\right] \\
&\quad + \left[f\left(\xi_1 + a \wedge B_1 + b \wedge A_1, \ldots, \xi_m + a \wedge B_m + b \wedge A_m\right) - f(\boldsymbol{\xi})\right] \\
&\qquad = \left[f\left(\xi_1 + a \wedge A_1, \xi_2, \ldots, \xi_{m-1}, \xi_m + a \wedge A_m\right) - f(\boldsymbol{\xi})\right] \\
&\qquad\qquad + \left[f\left(\xi_1 + b \wedge B_1, \ldots, \xi_m + b \wedge B_m\right) - f(\boldsymbol{\xi})\right] \\
&\qquad\qquad\quad + \left[f\left(\xi_1 + a \wedge B_1, \ldots, \xi_p + a \wedge B_p\right) - f(\boldsymbol{\xi})\right] \\
&\qquad\qquad\qquad + \left[f\left(\xi_1 + b \wedge A_1, \ldots, \xi_p + b \wedge A_p\right) - f(\boldsymbol{\xi})\right].
\end{aligned} \tag{6}$$

Proof We divide the proof in two steps. The first step is analogous to Lemma 3.1 in [2]

Step 1 We first show that

$$\begin{aligned}
&f\left(\xi_1 + a \wedge A_1 + a \wedge B_1, \ldots, \xi_m + a \wedge A_m + a \wedge B_m\right) + f(\boldsymbol{\xi}) \\
&= f\left(\xi_1 + a \wedge A_1, \ldots, \xi_m + a \wedge A_m\right) + f\left(\xi_1 + a \wedge B_1, \ldots, \xi_m + a \wedge B_m\right).
\end{aligned}$$

Since f is vectorially ext. one affine, for any $t \neq 0$, we have,

$$
\begin{aligned}
& f\left(\xi_1 + ta \wedge A_1 + a \wedge B_1, \ldots, \xi_m + ta \wedge A_m + a \wedge B_m\right) \\
& = f\left(\xi_1 + ta \wedge \left[A_1 + \frac{1}{t} B_1\right], \ldots, \xi_m + ta \wedge \left[A_m + \frac{1}{t} B_m\right]\right) \\
& = f(\boldsymbol{\xi}) + t\left[f\left(\xi_1 + ta \wedge \left[A_1 + \frac{1}{t} B_1\right], \ldots, \xi_m + ta \wedge \left[A_m + \frac{1}{t} B_m\right]\right) - f(\boldsymbol{\xi})\right] \\
& = f(\boldsymbol{\xi}) + t\left[f\left(\xi_1 + ta \wedge A_1, \ldots, \xi_m + ta \wedge A_m\right) + - f(\boldsymbol{\xi})\right] \\
& \qquad + f\left(\xi_1 + a \wedge \left[A_1 + B_1\right], \ldots, \xi_m + a \wedge \left[A_m + B_m\right]\right) \\
& \qquad\qquad - f\left(\xi_1 + ta \wedge A_1, \ldots, \xi_m + ta \wedge A_m\right).
\end{aligned}
$$

Letting $t \to 0$, we obtain the claim.

Step 2 Now by Step 1, we obtain,

$$
\begin{aligned}
& f\left(\xi_1 + a \wedge \left[A_1 + B_1\right], \ldots, \xi_m + a \wedge \left[A_m + B_m\right]\right) + f(\boldsymbol{\xi}) \\
& = f\left(\xi_1 + a \wedge A_1, \ldots, \xi_m + a \wedge A_m\right) + f\left(\xi_1 + a \wedge B_1, \ldots, \xi_m + a \wedge B_m\right).
\end{aligned}
$$

and

$$
\begin{aligned}
& f\left(\xi_1 + b \wedge \left[A_1 + B_1\right], \ldots, \xi_m + b \wedge \left[A_m + B_m\right]\right) + f(\boldsymbol{\xi}) \\
& = f\left(\xi_1 + a \wedge A_1, \ldots, \xi_m + a \wedge A_m\right) + f\left(\xi_1 + b \wedge B_1, \ldots, \xi_m + b \wedge B_m\right).
\end{aligned}
$$

But it is easy to see that

$$
\begin{aligned}
& f\left(\xi_1 + a \wedge \left[A_1 + B_1\right] + (b-a) \wedge B_1, \ldots, \xi_m + a \wedge \left[A_m + B_m\right] + (b-a) \wedge B_m\right) \\
& \qquad = f\left(\xi_1 + a \wedge A_1 + b \wedge B_1, \ldots, \xi_m + a \wedge A_m + b \wedge B_m\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& f\left(\xi_1 + a \wedge \left[A_1 + B_1\right] + (b-a) \wedge A_1, \ldots, \xi_m + a \wedge \left[A_m + B_m\right] + (b-a) \wedge A_m\right) \\
& \qquad = f\left(\xi_1 + a \wedge B_1 + b \wedge A_1, \ldots, \xi_m + a \wedge B_m + b \wedge A_m\right)
\end{aligned}
$$

Now we add the last two identities. Note that by Step 1, the sum of the LHS of the last two identities is

$$
\begin{aligned}
& f\left(\xi_1 + (a + (b-a)) \wedge \left[A_1 + B_1\right], \ldots, \xi_m + (a + (b-a)) \wedge \left[A_m + B_m\right]\right) \\
& \qquad + f\left(\xi_1 + a \wedge \left[A_1 + B_1\right], \ldots, \xi_m + a \wedge \left[A_m + B_m\right]\right).
\end{aligned}
$$

This coupled with the first two identities in Step 2 yields the result. ■

1 Notations

We gather here the notations which we use throughout this note. We reserve boldface english or greek letters to denote m-tuples of integers, real numbers, exterior forms etc as explained below.

Let $m, n \geq 1$ be integers.

- $\wedge$, $\lrcorner$, $\langle\ ,\ \rangle$ and $*$ denote the exterior product, the interior product, the scalar product and the Hodge star operator , respectively.

- $\boldsymbol{k}$ stands for an m-tuple of integers, $\boldsymbol{k} = (k_1, \ldots, k_m)$, where $1 \leq k_i \leq n$ for all $1 \leq i \leq m$, where $m \geq 1$ is a positive integer. We write $\boldsymbol{\Lambda}^{\boldsymbol{k}}(\mathbb{R}^n)$ (or simply $\boldsymbol{\Lambda}^{\boldsymbol{k}}$) to denote the Cartesian product $\displaystyle\prod_{i=1}^{m} \Lambda^{k_i}(\mathbb{R}^n)$, where $\Lambda^{k_i}(\mathbb{R}^n)$ denotes the vector space of all alternating k_i-linear maps $f : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k_i\text{-times}} \to \mathbb{R}$. For any integer r, we also employ the shorthand $\boldsymbol{\Lambda}^{\boldsymbol{k}+\boldsymbol{r}}$ to stand for the product $\displaystyle\prod_{i=1}^{m} \Lambda^{k_i+r}(\mathbb{R}^n)$. We denote elements of $\boldsymbol{\Lambda}^{\boldsymbol{k}}$ by boldface greek letters, except $\boldsymbol{\alpha}$, which we reserve for multiindices (see below). For example, we write $\boldsymbol{\xi} \in \boldsymbol{\Lambda}^{\boldsymbol{k}}$ to mean $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_m)$ is an m-tuple of exterior forms, with $\xi_i \in \Lambda^{k_i}(\mathbb{R}^n)$ for all $1 \leq i \leq m$. We also write $|\boldsymbol{\xi}| = \left(\displaystyle\sum_{i=1}^{m} |\xi_i|^2 \right)^{\frac{1}{2}}$. In general, boldface greek letters always mean an m-tuple of the concerned objects.

- If $\mathbf{k}$ is an m-tuple as defined above, we reserve the boldface greek letter $\boldsymbol{\alpha}$ for a multiindex, i.e an m-tuple of integers $(\alpha_1, \ldots, \alpha_m)$ with $0 \leq \alpha_i \leq \left[\frac{n}{k_i}\right]$ for all $1 \leq i \leq m$. We write $|\boldsymbol{\alpha}|$ and $|\boldsymbol{k\alpha}|$ for the sums $\displaystyle\sum_{i=1}^{m} \alpha_i$ and $\displaystyle\sum_{i=1}^{m} k_i \alpha_i$, respectively.

- For any $\mathbf{k}$ and $\boldsymbol{\alpha}$, as defined above, such that $1 \leq |\boldsymbol{k\alpha}| \leq n$, we write $\boldsymbol{\xi}^{\boldsymbol{\alpha}}$ for the wedge product

$$\xi_1^{\alpha_1} \wedge \ldots \wedge \xi_m^{\alpha_m} = \underbrace{\xi_1 \wedge \cdots \wedge \xi_1}_{\alpha_1\text{-times}} \wedge \ldots \wedge \underbrace{\xi_m \wedge \cdots \wedge \xi_m}_{\alpha_m\text{-times}} \in \Lambda^{|\boldsymbol{k\alpha}|}(\mathbb{R}^n).$$

  Clearly, if $\alpha_i = 0$ for some $1 \leq i \leq m$, ξ_i is absent from the product.

- Let $\mathbf{k}$ and $\boldsymbol{\alpha}$ be as defined above. Then for any $\boldsymbol{\xi} \in \boldsymbol{\Lambda}^{\boldsymbol{k}}$ and for any integer $1 \leq s \leq n$, $T_s(\boldsymbol{\xi})$ stands for the vector with components $\boldsymbol{\xi}^{\boldsymbol{\alpha}}$, where $\boldsymbol{\alpha}$ varies over all possible choices such that $|\boldsymbol{\alpha}| = s$, as long as there is at least one such non-trivial wedge power. As an example, if $m = 3$, then we immediately see that

$$\begin{aligned} T_1(\boldsymbol{\xi}) &= (\xi_1, \xi_2, \xi_3), \\ T_2(\boldsymbol{\xi}) &= \left(\xi_1^2, \xi_1 \wedge \xi_2, \xi_1 \wedge \xi_3, \xi_2^2, \xi_2 \wedge \xi_3, \xi_3^2\right) \text{ etc.} \end{aligned}$$

  $N(\boldsymbol{k})$ stands for the largest integer s for which there is at least one such non-trivial wedge power, i.e

$$N(\boldsymbol{k}) = \max\Big\{s \in \mathbb{N} : \exists \boldsymbol{\alpha} \text{ with } |\boldsymbol{\alpha}| = s \text{ such that } \boldsymbol{\xi}^{\boldsymbol{\alpha}} \neq 0 \\ \text{for some } \boldsymbol{\xi} \in \boldsymbol{\Lambda}^{\boldsymbol{k}}\Big\}.$$

$T(\boldsymbol{\xi})$ stands for the vector $T(\boldsymbol{\xi}) = \big(T_1(\boldsymbol{\xi}), \ldots, T_{N(\boldsymbol{k})}(\boldsymbol{\xi})\big)$, whose number of components is denoted by $\tau(n, \boldsymbol{k})$, i.e $T(\boldsymbol{\xi}) \in \mathbb{R}^{\tau(n,\boldsymbol{k})}$.

- *(i)* For $1 \leqslant k \leqslant n$, we write

  $$\mathcal{T}^k = \{(i_1, \cdots, i_k) \in \mathbb{N}^k : 1 \leqslant i_1 < \cdots < i_k \leqslant n\}.$$

  For $I = (i_1, \cdots, i_k) \in \mathcal{T}^k$, we write dx^I to denote $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$.

  (ii) For $i \in I$, we write $I_{\widehat{i}} = (i_1, \cdots, \widehat{i}, \cdots, i_k)$, where $\widehat{i}$ denotes the absence of the named index i. Note that, $I_{\widehat{i_p}} \in \mathcal{T}^{k-1}$, for all $1 \leqslant p \leqslant k$. Similarly, for $i, j \in I$, $i < j$, we write $I_{\widehat{ij}} = (i_1, \cdots, \widehat{i}, \cdots, \widehat{j}, \cdots, i_k)$.

  (iii) Given $I \in \mathcal{T}^{k_1}$ and $J \in \mathcal{T}^{k_2}$ with $I \cap J = \emptyset$, i.e I and J have no common index, we write $[IJ]$ to denote the increasing multiindex formed by the the indices in I and J. In other words $[IJ]$ is the permutation of the indices such that $[IJ] \in \mathcal{T}^{k_1+k_2}$. Furthermore, we define the sign of $[I, J]$, denoted by $\operatorname{sgn}[I, J]$, as

  $$dx^{[IJ]} = \operatorname{sgn}[I, J]\, dx^I \wedge dx^J.$$

- We shall need one particular permutation operation of indices.

  Definition 4 (1-flip) *Let $l, s \geq 1$, let $J \in \mathcal{T}^s$, $I \in \mathcal{T}^l$ be written as, $J = \{j_1 \ldots j_s\}$, $I = \{i_1 \ldots i_l\}$ with $I \cap J = \emptyset$. Let $\tilde{J} \in \mathcal{T}^s$, $\tilde{I} \in \mathcal{T}^l$. We say that the pair $(\tilde{I}, \tilde{J})$ is obtained from the pair (I, J) by a 1-flip interchanging j_p with i_m, for some $1 \leq p \leq s$, $1 \leq m \leq l$, if*

  $$\tilde{J} = [j_1 \ldots j_{p-1} i_m j_{p+1} \ldots j_s] \text{ and } \tilde{I} = [i_1 \ldots i_{m-1} j_p i_{m+1} \ldots i_l],$$

  where the square brackets mean the increasing multiindex formed by the indices inside the brackets.

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